SECURITY CLASSIFICATION OF THIS PAGE

AD-A199 988

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

DTIC ELECTE OCT 0 6 1988 S D

(2)

1b. RESTRICTIVE MARKINGS

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE

3. DISTRIBUTION / AVAILABILITY OF REPORT
Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)
3

5. MONITORING ORGANIZATION REPORT NUMBER(S)
AFOSR-TR- 88-1014

6a. NAME OF PERFORMING ORGANIZATION
Clarkson University

6b. OFFICE SYMBOL (If applicable)

7a. NAME OF MONITORING ORGANIZATION
AFOSR

6c. ADDRESS (City, State, and ZIP Code)
Division of Research
Potsdam, NY 13676

7b. ADDRESS (City, State, and ZIP Code)
Bldg 410
BAFB DC 20332-6448

8a. NAME OF FUNDING / SPONSORING ORGANIZATION Air Force Office of Scientific Research

8b. OFFICE SYMBOL (If applicable)
NM

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
AFOSR-87-0310

8c. ADDRESS (City, State, and ZIP Code)
Building 410
Bolling AFB, DC 20332-6448

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO	WORK UNIT ACCESSION NO
61102F	2304	A4	

11. TITLE (Include Security Classification) Collective Properties of Neural Systems and Their Relation to Other Physical Models

12. PERSONAL AUTHOR(S)
Barouch, Eytan; Fokas, A.

13a. TYPE OF REPORT
Final

13b. TIME COVERED
FROM 870701 TO 880831

14. DATE OF REPORT (Year, Month, Day)
880805

15. PAGE COUNT

16. SUPPLEMENTARY NOTATION
375-642

17. COSATI CODES

FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

LIST OF PUBLICATIONS

1. H. Araki and E. Barouch, Lett. Math. Phys. **14**, 227-234, (1987).
2. S.V. Babu, E. Barouch, and B. Bradie, J. Vac. Sci. Tech., **B6** 2, 564-568, (1988).
3. A.S. Fokas, P.M. Santini: Recursion Operators and Bi-Hamiltonian Structures in Multidimensions II, Commun. in Math. Phys. **116**, 449-474 (1988).
4. P.M. Santini and A.S. Fokas, Comm. Math. Phys. **115**, 375-419 (1988).
5. A.S. Fokas, P.M. Santini: Bi-Hamiltonian Formulation of the Kadomtsev- Petviashvili and Benjamin-Ono Equations, J. Math. Phys., **29** (3) 604-617 (1988).

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION
202 767-5025

22a. NAME OF RESPONSIBLE INDIVIDUAL
~~Eytan Barouch~~ Arje Nachman

22b. TELEPHONE (Include Area Code)
~~(315) 268-2371~~

22c. OFFICE SYMBOL
NM

DD Form 1473, JUN 86 Previous editions are obsolete. SECURITY CLASSIFICATION OF THIS PAGE

88 10 5 295

6. E. Barouch, A.S. Fokas, and V. Papageorgiou, Algorithmic Construction of the Recursion Operators of Toda and Landau-Lifshitz Equation, INS #90, RIMS Publications, **650**, 179-196, (1988).

7. E. Barouch, A.S. Fokas, and V. Papageorgiou, The BiHamiltonian Formulation of the Landau-Lifshitz Equation, INS #89, February 1988to appear in J. of Math. Phys.

8. E. Barouch, B.D. Bradie, and S.V. Babu, Resist Development Described by Least Action Principle-Line Profile Prediction, preprint, 1988.

9. A.S. Fokas: An Initial-Boundary Value Problems for a Class of Nonlinear Schrödinger Equations, INS #81, January 1988, to appear in Physica D.

COLLECTIVE PROPERTIES OF NEURAL SYSTEMS AND
THEIR RELATION TO OTHER PHYSICAL MODELS

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August 5, 1988

Final Report for July 1, 1987 to August 31, 1988

J

Prepared for:
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
Building 410
Bolling AFB, DC 20332-6448

A-1

AFOSR-TR- 88-1014

AFOSR-87-0310 - Neural Systems

FINAL REPORT ON 375-642

During the tenure of this contract we have made progress on three fronts:

1. The recursion operator of the Landau-Lifshitz equation has been computed explicitly. This has been achieved algorithmically by utilizing methods introduced earlier. It should be emphasized that in addition to the important implications of these results to general lattice theories and neural networks, the answers obtained are novel on their own merit since textbooks referred to constructing the above recursion operation as an outstanding open problem.

We have been invited to lecture on the above work in several major international conferences (Italy, Japan, South America, France, Canada, US).

2. We have continued our study of nonlinear optics. We have introduced a new system of nonlinear PDE's that governs the development path of photoresist fabrication. We have employed a proof given in collaboration with Araki concerning an iteration scheme, used throughout the analysis. We have reported this work in various publications and in a number of international conferences.

3. Substantial progress has been made towards solving the nonlinear Schrödinger (NLS) equation on the half-line. Finite boundedness in conjuction with nonlinear evolution equations have alluded investigators for years. Since nonlinear optics is to be employed on finite boundaries, a major thrust was needed to achieve viable results. A new method has been introduced and tested on the NLS on the half-line. For the first time concrete analytical results have been obtained, and the entire problem has been reduced to linearizing a certain equation satisfied by the scattering data. This linearization and the application of the above method to other important evolution equaitons is under investigation.

LIST OF PUBLICATIONS

1. H. Araki and E. Barouch, Lett. Math. Phys. **14**, 227-234, (1987).

2. S.V. Babu, E. Barouch, and B. Bradie, J. Vac. Sci. Tech., **B6** 2, 564-568, (1988).

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Letters in Mathematical Physics **14** (1987) 227–234.

A Note on an Exact Solution for the Optical Absorbance by Thin Films*

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(Received: 1 July 1987)

Abstract. The Babu–Barouch solution of Berning's difference equation for the electromagnetic fields within optical thin films is shown to converge in the continuum limit to a solution (expressed as a converging series) of the limiting differential equation.

1. Introduction

In nonlinear optics, there is mounting interest in a deeper analysis of the effect of the nonlinear interplay between the light intensity and the complex refractive index. A basic formulation for the electromagnetic field in thin films was introduced by Berning [2]. Recently, Babu and Barouch [1] obtained an exact analytical solution of Berning's difference equations in a closed form. This difference equation describes the situation where the thin film is divided into many sublayers and all relevant quantities in each layer are constant within the layer.

The purpose of this Letter is to discuss the continuum limit, i.e., the limit of the width of each sublayer converging to 0. We will establish mathematically that the Babu–Barouch expression for the electromagnetic field converges in this limit to a unique solution (explicitly given by a converging series) of differential (or equivalently integral) equations which is a natural limit of Berning's difference equations.

2. Results

Let E_j and H_j be the electric and magnetic fields in the jth sublayer, λ be the wavelength of the incident (exposing) beam, l_j be the thickness of the jth layer,

$$N_j = n_j - iK_j \tag{2.1}$$

* Supported in part by the NSF Grant #ECS 8611298 and the mathematics division of AFOSR.

be its complex refractive index and

$$\phi_j = 2\pi l_j N_j/\lambda \tag{2.2}$$

(the phase thickness of the jth layer).

Then the Babu–Barouch solution is

$$\begin{aligned} E_j = \int_0^1 \mathrm{d}x \Bigg\{ &(E_m \, \mathrm{e}^{-2\pi i x 2^m} + H_m) \, [\mathrm{e}^{2\pi i x 2^{j+1}} \cos\phi_{j+1} + (i/N_{j+1}) \sin\phi_{j+1}] \times \\ &\times \Bigg[\prod_{l=j+1}^{m-1} \{[\mathrm{e}^{2\pi i x 2^l} + 1] \cos\phi_{l+1} + [(i/N_{l+1}) \, \mathrm{e}^{-2\pi i x 2^l} + \\ &+ iN_{l+1} \, \mathrm{e}^{2\pi i x 2^{l+1}}] \sin\phi_{l+1}\} \Bigg] \Bigg\} , \end{aligned} \tag{2.3}$$

$$\begin{aligned} H_j = \int_0^1 \mathrm{d}x \Bigg\{ &(E_m \, \mathrm{e}^{-2\pi i x 2^m} + H_m) \, [\cos\phi_{j+1} + iN_{j+1} \, \mathrm{e}^{2\pi i x 2^{j+1}} \sin\phi_{j+1}] \times \\ &\times \Bigg[\prod_{l=j+1}^{m-1} \{[\mathrm{e}^{2\pi i x 2^l} + 1] \cos\phi_{l+1} + [(i/N_{l+1}) \, \mathrm{e}^{-2\pi i x 2^l} + \\ &+ iN_{l+1} \, \mathrm{e}^{2\pi i x 2^{l+1}}] \sin\phi_{l+1}\} \Bigg] \Bigg\} . \end{aligned} \tag{2.4}$$

We will consider the limit of

$$m \to \infty , \qquad \delta \equiv \max_j (l_j) \to 0 , \tag{2.5}$$

$$\sum_{j=1}^{m} l_j = D \quad \text{(the total thickness of the film).} \tag{2.6}$$

We assume that there is a smooth function $N(z)$ such that

$$N_j = N\left(\sum_{k=1}^{j} l_k \right) . \tag{2.7}$$

For $j(m)$ such that

$$\lim \sum_{j=1}^{j(m)} l_j = z , \tag{2.8}$$

we prove that the limits

$$E(z) = \lim E_{j(m)} , \qquad H(z) = \lim H_{j(m)} \tag{2.9}$$

exist and can be expressed as a series of multiple integral expressions (see equations (4.15)–(4.20)). They are the unique solution of coupled integral equations (5.9) and (5.10). They are also the unique solution of the coupled differential equations (5.11) and (5.12) with $E(D)$ and $H(D)$ as initial conditions.

3. x-integration

Each factor in (2.3) and (2.4) is a sum of two terms (for the first two factors) or four terms (for the remaining factors), each term proportional to an integer power of $w = e^{2\pi i x}$. After multiplying out into a Laurent polynomial of w, the x-integration eliminates all terms except those independent of w.

Case (1): $E_m = 1$, $H_m = 0$.
The negative (-2^mth) power of w multiplying E_m in the first factor can be cancelled out by multiplying the first terms of all remaining factors: $2^{j+1} + \Sigma 2^l = 2^m$. Instead of taking the first term from all factors, we may replace some of them by other terms. Replacement by the second or third term will decrease the power of w by 2^l or 2^{l+1}, respectively. Replacement by the fourth term will increase it by 2^l $(2^{l+1} = 2^l + 2^l)$. These changes of the power of w are listed in Table I.

Table I. Change of powers of w according to the chosen terms – Case (1)

	2nd term	3rd term	4th term
2nd factor	-2^{j+1}	–	–
$l = j+1$	-2^{j+1}	-2^{j+2}	2^{j+1}
$l = j+2$	-2^{j+2}	-2^{j+3}	2^{j+2}
...	...	...	...
l-factor	-2^l	-2^{l+1}	2^l
...	...	...	...

For obtaining the w-independent product, we have to balance the decrease and increase. If the fourth term is chosen in the $l = k$ factor with an increase of 2^k, this can be cancelled out only by one of the following combinations: The second term from the l-factors with $l = k-1, \ldots, k'$ $(j < k' \leqslant k$ and none here if $k' = k)$ and the third term from the l-factor for $l = k'-1$ if $k' > j+1$; the second term from the second factor if $k' = j+1$.

The above type of sequence of choices may be repeated in mutually nonoverlapping sequences of factors. Thus, E_j of (2.3) for $E_m = 1$, $H_m = 0$ is given by

$$E_j = \left(\prod_{l=j+1}^{m} \cos\phi_l\right) \sum (-1)^n \prod_{\nu=1}^{n} (N_{k_\nu+1}/N_{k'_\nu}) \tan\phi_{k'_\nu} \tan\phi_{k_\nu+1} \tag{3.1}$$

where the sum is over $n = 0, 1, 2, \ldots$ and, for $n > 0$, over all possible integers $k_1 \ldots k_n$, $k'_1, \ldots, k'_n$ satisfying

$$m > k_1 > k'_1 - 1 > k_2 > k'_2 - 1 > \ldots > k_n > k'_n - 1 \geqslant j\,. \tag{3.2}$$

The same reasoning gives the following expression for H_j:

$$H_j = (iN_{j+1} \sin\phi_{j+1}) \left(\prod_{l=j+2}^{m} \cos\phi_l\right) \times$$

$$\times\, \Sigma^{(1)}(-1)^n \prod_{v=1}^{n} (N_{k_v+1}/N_{k'_v}) \tan\phi_{k'_v} \tan\phi_{k_v+1} +$$

$$+\left(\prod_{l=j+1}^{m} \cos\phi_l\right) \Sigma^{(2)}(-1)^{n-1} iN_{k_n+1} \tan\phi_{k_n+1} \times$$

$$\times \prod_{v=1}^{n-1} (N_{k_v+1}/N_{k'_v}) \tan\phi_{k'_v} \tan\phi_{k_v+1}, \tag{3.3}$$

where the sum in (3.2) is now divided into two parts $\Sigma^{(1)}$ and $\Sigma^{(2)}$ according to $k'_n \neq j+1$ or $k'_n = j+1$ ($n = 0$ term is in $\Sigma^{(1)}$).

Case (2): $E_m = 0$, $H_m = 1$.
We follow the same method of computation as in the previous case. In (2.4) for H_j, the product of the x-independent term (the first term in the second factor and the second term in the remaining factors) is taken as the standard for measuring the increase or decrease of powers of w according to the choice of terms in each factor, which is listed in Table II. Thus, we obtain

$$H_j = \left(\prod_{l=j+1}^{m} \cos\phi_l\right) \sum(-1)^n \prod_{v=1}^{n} (N_{k'_v}/N_{k_v+1}) \tan\phi_{k'_v} \tan\phi_{k_v+1} \tag{3.4}$$

where the summation is the same as in (3.1). We also obtain

$$E_j = (i/N_{j+1}) \sin\phi_{j+1} \left(\prod_{l=j+2}^{m} \cos\phi_l\right) \times$$

$$\times\, \Sigma^{(1)}(-1)^n \prod_{v=1}^{n} (N_{k'_v}/N_{k_v+1}) \tan\phi_{k'_v} \tan\phi_{k_v+1} +$$

$$+\left(\prod_{l=j+1}^{m} \cos\phi_l\right) \Sigma^{(2)}(-1)^{n-1} (i/N_{k_n+1}) \tan\phi_{k_n+1} \times$$

$$\times \prod_{v-1}^{n-1} (N_{k'_v}/N_{k_v+1}) \tan\phi_{k'_v} \tan\phi_{k_v+1}, \tag{3.5}$$

where the summation is the same as in (3.3).

Table II. Change of powers of w according to the chosen terms – Case (2)

	1st term	2nd term	3rd term	4th term
2nd factor	0	2^{j+1}	–	–
$l = j+1$	2^{j+1}	0	-2^{j+1}	2^{j+2}
$l = j+2$	2^{j+2}	0	-2^{j+2}	2^{j+3}
...	...	...	...	...
l-factor	2^l	0	-2^l	2^{l+1}
...	...	...	...	...

4. Continuum Limit

We first discuss the limit of (3.1) under (2.5) and (2.6). We do this by showing that (i) the product of $\cos\phi_l$ converges to 1, (ii) the sum is absolutely convergent with the convergence uniform in m and $\{l_j\}$, and (iii) the sum for each fixed n converges to an explicit multiple integral expression.

(i) Let

$$G = \sup_{0 \leqslant z \leqslant D} \max\{\,|N(z)|, |N(z)|^{-1}\} \quad (<\infty)\,. \tag{4.1}$$

Due to $1 \geqslant \cos\phi \geqslant 1 - (\phi^2/2)$, we obtain for sufficiently small δ (so that $|\phi_l| \leqslant 2\pi G\delta/\lambda < 1$),

$$1 \geqslant \prod_{l=j+1}^{m} \cos\phi_l \geqslant \prod_{k=1}^{m} (1 - 2\pi^2 G^2 \lambda^{-2} l_k^2)\,. \tag{4.2}$$

Since

$$\sum_{k=1}^{m} l_k^2 \leqslant \delta \sum_{k=1}^{m} l_k = \delta D \tag{4.3}$$

tends to 0, the extreme right-hand side of (4.2) tends to 1 as $\delta \to 0$. Therefore

$$\lim \prod_{l=j+1}^{m} \cos\phi_l = 1\,. \tag{4.4}$$

(ii) For sufficiently small δ so that $|\phi_l| \leqslant \pi/3$, we have $|\cos\phi_l| \geqslant \cos\pi/3 = \frac{1}{2}$ and hence $|\tan\phi_l| \leqslant 2|\sin\phi_l| \leqslant 2|\phi_l|$. By (2.2), each term in the sum in (3.1) is majorized by

$$(4G^2)^n \prod_{\nu=1}^{n} (|\phi_{k'_\nu}|\,|\phi_{k_\nu+1}|) = (4\pi G^2/\lambda)^{2n} \prod_{\nu=1}^{n} (l_{k'_\nu} l_{k_\nu+1})\,. \tag{4.5}$$

Therefore, the sum in (3.1) is majorized by

$$\sum_{n=0}^{\infty} (4\pi G^2/\lambda)^{2n} (\Sigma l_{k_1+1} l_{k'_1} \cdots l_{k_n+1} l_{k'_n}) \tag{4.6}$$

where the second sum is over all integers $k_1, k'_1, \ldots, k'_n$ such that

$$m \geqslant k_1 + 1 > k'_1 > k_2 + 1 > \ldots > k_n + 1 > k'_n > j\,. \tag{4.7}$$

The permutation of these integers produces disjoint sets of the ordered set of $2n$ indices. Therefore, the sum is majorized by

$$\sum_{n=0}^{\infty} (4\pi G^2/\lambda)^{2n} (2n)!^{-1} \Sigma (l_{k_1+1} l_{k'_1} \cdots l_{k_n+1} l_{k'_n}) \tag{4.8}$$

where the sum is now all integers such that $m \geqslant k_i + 1 > j$ and $m \geqslant k'_i > j$. Since

$$\sum_{k=j+1}^{m} l_k \leqslant \sum_{k=1}^{m} l_k = D\,, \tag{4.9}$$

we have now the majorization of (3.1) by

$$\sum_{n=0}^{\infty}(2n)!^{-1}(4\pi G^2D/\lambda)^{2n} = \cosh(4\pi G^2D/\lambda) < \infty . \tag{4.10}$$

(iii) For sufficiently small δ so that $|\phi_k| \leqslant \pi/3$, we obtain

$$|\tan\phi_k - \phi_k| \leqslant (\sec^2\phi_k)|\phi_k|^3 \leqslant 4|\phi_k|^3 \leqslant C\delta^2|\phi_k| \tag{4.11}$$

by the mean value theorem (and the monotonicity of $\sec\phi_k$ for small ϕ_k). By combining this estimate with the estimate leading to (4.10), we obtain

$$\begin{aligned}\sum&\left|\left\{\prod_{\nu=1}^{n}(N_{k_\nu+1}/N_{k_\nu})(\tan\phi_{k_\nu}\tan\phi_{k_\nu+1}) - \prod_{\nu=1}^{n}(N_{k_\nu+1}/N_{k_\nu})(\phi_{k_\nu}\phi_{k_\nu+1})\right\}\right| \\ &\leqslant C^2\sum_{n=1}^{\infty}(2n)!^{-1}(2n)(4\pi G^2D/\lambda)^{2n} \\ &= (4C\pi G^2D\delta^2/\lambda)\sinh(4\pi G^2D/\lambda)\end{aligned} \tag{4.12}$$

which tends to 0 as $\delta \to 0$. On the other hand, the summation

$$\begin{aligned}\sum&\prod_{\nu=1}^{n}(N_{k_\nu+1}/N_{k_\nu})(\phi_{k_\nu}\phi_{k_\nu+1}) \\ &= (2\pi/\lambda)^{2n}\sum\prod_{\nu=1}^{n}N\left(\sum_{k=1}^{k_\nu+1}l_k\right)^2(l_{k_\nu}l_{k_\nu+1}),\end{aligned} \tag{4.13}$$

(see (2.7)) with n fixed and (3.2) satisfied, tends to

$$\begin{aligned}(2\pi/\lambda)^{2n}&\int_z^D dz_n'\int_{z_n'}^D N(z_n)^2\,dz_n\int_{z_n}^D\cdots\int_{z_1'}^D N(z_1)^2\,dz_1 \\ &= (2\pi/\lambda)^{2n}\int_z^D N(z_1)^2\,dz_1\int_z^{z_1}dz_1'\int_z^{z_1'}\cdots\int_z^{z_{n-1}'}N(z_n)^2\,dz_n\int_z^{z_n}dz_n'\end{aligned} \tag{4.14}$$

as $\Sigma_{k=1}^{l}l_k \to z$ by the definition of the (Riemann) integral.

Thus, we have established the following when $E_m = 1$, $H_m = 0$.

$$\begin{aligned}\lim E_j &= \sum_{n=0}^{\infty}(-1)^n(2\pi/\lambda)^{2n}\int_z^D dz_n'\int_{z_n}^D N(z_n)^2\,dz_n\cdots\int_{z_1'}^D N(z_1)^2\,dz_1 \\ &= \sum_{n=0}^{\infty}(-1)^n(2\pi/\lambda)^{2n}\int_z^D N(z_1)^2\,dz_1\int_z^{z_1}dz_1'\cdots\int_z^{z_{n-1}'}N(z_n)^2\,dz_n\int_z^{z_n}dz_n' \\ &\equiv E_e(z).\end{aligned} \tag{4.15}$$

In (3.3), the first term tends to 0 because $N_{l+1}\sin\phi_{l+1} \to 0$ while the rest is estimated by (4.10). Therefore, we obtain

$$\lim H_j = \sum_{n=0}^{\infty}i(-1)^n(2\pi/\lambda)^{2n+1}\int_z^D N(z_{n+1})^2\,dz_{n+1}\int_{z_{n+1}}^D dz_n'\cdots\int_{z_1'}^D N(z_1)^2\,dz_1$$

$$= \sum_{n=0}^{\infty} i(-1)^n(2\pi/\lambda)^{2n+1} \int_z^D N(z_1)^2\,dz_1 \int_z^{z_1} dz_1' \ldots \int_z^{z_n} dz_n' \int_z^{z_n'} N(z_{n+1})^2\,dz_{n+1}$$

$$\equiv H_e(z) \tag{4.16}$$

when $E_m = 1$, $H_m = 0$.

Similarly, when $E_m = 0$, $H_m = 1$, we obtain

$$\lim H_j = \sum_{n=0}^{\infty} (-1)^n(2\pi/\lambda)^{2n} \int_z^D N(z_n')^2\,dz_n' \int_{z_n'}^D dz_n \ldots \int_{z_2}^D N(z_1')^2\,dz_1' \int_{z_1'}^D dz_1$$

$$= \sum_{n=0}^{\infty} (-1)^n(2\pi/\lambda)^{2n} \int_z^D dz_1 \int_z^{z_1} N(z_1')^2\,dz_1' \ldots \int_z^{z_n} N(z_n')^2\,dz_n'$$

$$\equiv H_h(z)\,, \tag{4.17}$$

$$\lim E_j = \sum_{n=0}^{\infty} (i(-1)^n(2\pi/\lambda)^{2n+1} \int_z^D dz_{n+1} \int_{z_{n+1}}^D N(z_n')^2\,dz_n' \ldots \int_{z_2}^D N(z_1)^2\,dz_1' \int_{z_1'}^D dz_1$$

$$= \sum_{n=0}^{\infty} i(-1)^n(2\pi/\lambda)^{2n+1} \int_z^D dz_1 \int_z^{z_1} N(z_1')^2\,dz_1' \ldots \int_z^{z_n} N(z_n')^2\,dz_n' \int_z^{z_n'} dz_{n+1}$$

$$\equiv E_h(z)\,. \tag{4.18}$$

The general case can then be obtained as linear combinations:

$$E(z) = E(D)E_e(z) + H(D)E_h(z)\,; \tag{4.19}$$

$$H(z) = E(D)H_e(z) + H(D)H_h(z)\,; \tag{4.20}$$

5. Integral and Differential Equations

From (4.16), we see that

$$H_e(z) = (2\pi i/\lambda) \int_z^D N(z_{n+1})^2 E_e(z_{n+1})\,dz_{n+1}\,, \tag{5.1}$$

$$(d/dz)H_e(z) = -(2\pi i/\lambda)N(z)^2 E_e(z)\,. \tag{5.2}$$

From (4.15), we also see that

$$E_e(z) = 1 + (2\pi i/\lambda) \int_z^D H_e(z_n')\,dz_n'\,, \tag{5.3}$$

$$(d/dz)E_e(z) = -(2\pi i/\lambda)H_e(z)\,. \tag{5.4}$$

Conversely, the coupled integral equations (5.1) and (5.3) for E_e and H_e can be solved by iteration, giving rise to the first expressions of (4.15) and (4.16) as the unique solution. The coupled differential equations (5.2) and (5.4) together with the boundary condition $E_e(D) = 1$, $H_e(D) = 0$ yield (5.1) and (5.3) and, hence, again have a unique set of solutions $E_e(z)$ and $H_e(z)$.

In case (2), we obtain the following from (4.17) and (4.18)

$$E_h(z) = (2\pi i/\lambda) \int_z^D H_h(z_{n+1})\, \mathrm{d}z_{n+1}, \tag{5.5}$$

$$(\mathrm{d}/\mathrm{d}z)E_h(z) = -(2\pi i/\lambda)H_h(z), \tag{5.6}$$

$$H_h(z) = 1 + (2\pi i/\lambda) \int_z^D N(z_n')^2 E_h(z_n')\, \mathrm{d}z_n', \tag{5.7}$$

$$(\mathrm{d}/\mathrm{d}z)H_h(z) = -(2\pi i/\lambda)N(z)^2 E_h(z). \tag{5.8}$$

Combining the two results, we obtain the following coupled integral and differential equations for the general case

$$E(z) = E(D) + (2\pi i/\lambda) \int_z^D H(z')\, \mathrm{d}z', \tag{5.9}$$

$$H(z) = H(D) + (2\pi i/\lambda) \int_z^D N(z')^2 E(z')\, \mathrm{d}z', \tag{5.10}$$

$$(\mathrm{d}/\mathrm{d}z)E(z) = -(2\pi i/\lambda)H(z), \tag{5.11}$$

$$(\mathrm{d}/\mathrm{d}z)H(z) = -(2\pi i/\lambda)N(z)^2 E(z). \tag{5.12}$$

Equations (2.3) and (2.4) are solutions of the difference equation (1) of [1]. Differential equations (5.11) and (5.12) are the continuum limit of this difference equation. Thus we have shown that the limit of the solution (of (1) in [1]) is the solution of the limit of the equation.

Acknowledgement

One of us (H. A.) would like to express his appreciation and gratitude for hospitality and financial support at the Department of Mathematics and Computer Science, Clarkson University, where this work was collaborated.

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Calculation of image profiles for contrast enhanced lithography[a)]

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(Received 20 July 1987; accepted 20 October 1987)

Simultaneous bleaching of a contrast enhancing film (CEF) and the underlying positive photoresist is considered in the absence of any interface or substrate reflectivity. The intensity transmitted by the CEF is determined as a function of exposure time exactly using the absorptivity of the film in Dill's model equations. Corresponding to this time dependent transmitted intensity, the concentration profiles in the positive photoresist have been expressed exactly in closed form. Relations, that implicitly define the developed image profile, are derived assuming that the resist development can be approximated by a two state process. Furthermore, they are solved numerically for a polysilane–AZ2400 resist system and a model CEM-388–resist combination proposed by Mack. The predicted image profiles are in excellent agreement with the experimentally determined profiles of Hofer *et al.*, for the polysilanes, and the predictions of PROLITH for the model system of Mack.

I. INTRODUCTION

In contrast enhanced lithography[1,2] (CEL) a conventional UV resist is coated with a thin bleachable contrast enhancing film (CEF) that exhibits "bleaching latency." [3] Exposure of the CEF above a certain threshold level results in increased transmission, while exposures below the threshold produce little change. Significant improvement in the quality of projection printed features has been reported by Griffing and West for 0.5 μm images[1,2] and by Hofer *et al.*[3] for 1.0 μm images using CEL. Griffing and West[1,2] used an undisclosed organic dye for the CEF, while Hofer *et al.*[3] used a 0.2-μm-thick aliphatic polysilane as the CEF. Hofer *et al.* also reported that the nonlinear bleaching of the polysilane film used by them was well described by an effective concentration dependent Dill's A parameter,[4] given by

$$A_{\rm eff} = [0.5 + 1.4(M_c - 0.4)]A\,, \tag{1}$$

where M_c is the concentration of the unbleached polysilane with absorbance A.

Recently Dill's model equations for the exposure, bleaching of "linear" resist materials, have been solved exactly in the absence of standing waves,[5] and the solution extended to the image reversal process with positive photoresist.[6] More recently, Dill's model equations have also been solved in a closed form when the bleaching characteristics are nonlinear.[7] It has been applied to the simultaneous bleaching of a positive resist and that of a contrast enhancing polysilane film, assuming that the reflections can be ignored. As Oldham argues[8] ignoring reflections "is actually appropriate in many cases since the CEL itself is a major aid in suppressing reflections." In any case, the effects of reflections from the interfaces can be included using the recently derived closed form solution for the optical absorbance of thin films in the presence of standing waves.[9] A comprehensive discussion of all reflections and standing waves in the CEL process will be presented in a later publication.

In this paper, using the closed form solution for the concentration of the photoactive compound (PAC) in the underlying photoresist film, an implicit functional relation for the developed image contour is derived. The derivation assumes, following Greeneich[10] and Watts,[11] that resist dissolution proceeds down to the substrate in the z direction first, followed by a lateral development in the x direction.[12] The final image profile is obtained for an AZ-2400 resist film exposed through a polysilane layer and developed in an AZ-2401 developer. The calculated images are in excellent agreement with the images reported by Hofer *et al.*[3] Developed image profiles have also been calculated for the model CEL-positive resist combination investigated by Mack[13] using PROLITH to simulate CEM-388. The two calculations are in good agreement.

II. BLEACHING OF THE CEF

First, the intensity transmitted by the CEF is determined as a function of position and time. The *simultaneous* bleaching of the underlying photoresist is determined as a function of position and exposure using this transmitted intensity.

Equation (1) may be rewritten more generally as

$$A_{\rm eff} = \alpha M_c + \beta \tag{2}$$

with the subscript c denoting the CEF and where α and β are two material dependent constants. For the CEM class of materials proposed by Griffing and West[1] and investigated by Mack,[13] α in Eq. (2) is equal to zero.

The bleaching of the CEF is described in terms of Dill's model equations by

$$\frac{\partial M_c}{\partial t} = -I_c M_c C_c \tag{3}$$

and

$$\frac{\partial I_c}{\partial z} = -(\alpha M_c^2 + \beta M_c + B_c)I_c, \tag{4}$$

with the initial and boundary conditions

$$M_c(z,x,0) = 1 \quad 0<z<1 \tag{5}$$

and

$$I_c(0,x,t) = I_0(x) \quad \text{for all } x \text{ and } t. \tag{6}$$

Here z is the normalized depth into the CEF measured from the top, x is a lateral coordinate measured across the image and used to define the incident aerial image intensity $I_0(x)$, B_c is the exposure independent absorption parameter,[4] C_c is the bleaching rate, and t is the exposure time.

Following Babu and Barouch,[5] a first integral for M_c can be written as

$$(\alpha/2)M_c^2 + \beta M_c + B_c \ln M_c + \frac{\partial}{\partial z}\ln M_c = f(z), \tag{7}$$

where $f(z)$ is an integration constant. Substitution of $t=0$ and use of Eq. (5) in Eq. (7) yields $f(z) = \alpha/2 + \beta$. Then Eq. (7) can be integrated as

$$z = \int_{g(x,t)}^{M_c(z,x,t)} d\ln y\{(\alpha/2)(1-y^2) + \beta(1-y) - B_c \ln y\}^{-1}. \tag{8}$$

The lower limit $g(x,t)$ is yet to be determined. It is obtained as $M_c(0,x,t)$ upon substituting $z=0$ in Eq. (8). But from Eqs. (3) and (6),

$$g(x) = M_c(0,x,t) = \exp[-I_0(x)C_c t]. \tag{9}$$

Differentiating Eq. (8) with respect to t, and combining it with Eq. (2), one also obtains

$$I_c(z,x,t) = \frac{I_0(x)[(\alpha/2)(1-M_c^2) + \beta(1-M_c) - B_c \ln M_c]}{(\alpha/2)[1-e^{-2I_0(x)C_c t}] + \beta[1-e^{-I_0(x)C_c t}] + B_c I_0(x) C_c t}. \tag{10}$$

The ratio $I_c(1,x,t)/I_0(x)$ is a measure of the improvement in the contrast of the aerial image due to the nonlinear bleaching of the CEF.

III. BLEACHING OF THE POSITIVE RESIST

Simultaneously, as the transmission of the CEF increases, bleaching of the PAC in the underlying positive resist continues. The bleaching of the PAC is described by

$$\frac{\partial M}{\partial t} = -IMC \tag{11}$$

and

$$\frac{\partial I}{\partial \delta} = -(AM+B)I \quad 0<\delta<1, \tag{12}$$

with A, B, C being the usual resist parameters and M the concentration of the PAC. A and B, as well as the depth parameter δ, are nondimensionalized using the photoresist thickness. Thus, $\delta=0$ at the CEF–resist interface and $\delta=1$ at the resist–substrate interface.

The initial condition is still given by

$$M(\delta,x,0) = 1, \tag{13}$$

but the boundary condition for $I(\delta,x,t)$ at $\delta=0$ is now time dependent due to the increased transmission of the CEF.

$$I(0,x,t) = I_c(1,x,t) \tag{14}$$

and is determined from Eq. (10).

However, this does not create any difficulty for solving Eqs. (11) and (12). Again following Babu and Barouch,[5] the solution $M(\delta,x,t)$ is determined implicitly in the absence of substrate reflectivity, as

$$\delta = \int_{h(x,t)}^{M(\delta,x,t)} d\ln y[A(1-y) - B\ln y]^{-1}. \tag{15}$$

The lower limit $h(x,t)$ is determined, as before, by substituting $\delta=0$ in Eq. (15) and then using Eq. (14) in Eq. (11),

$$h(x,t) = M(\delta=0,x,t) = \exp\left[-C\int_0^t I_c(1,x,t)dt\right]. \tag{16}$$

This completes the determination of the closed form solution of the PAC concentration profile in the CEL process, when interface reflectivities are not significant.

IV. CEL IMAGE PROFILE CALCULATION

The resist dissolution process can be approximately represented by a two-stage process.[10,11] In the first stage, dissolution proceeds in the z direction until all the resist is cleared from the substrate in the regions of maxima in the transmitted intensity. This is followed by dissolution in the lateral (x) direction till the end of the development process.

Let the total development time be t_d, and the phenomenological dissolution–development rate function be given by $R[M]$ (see Refs. 13 and 14). Then

$$t_d = t_\delta + t_x, \tag{17}$$

where

$$t_\delta = \int_0^{\delta(x)} \frac{d\delta'}{R[M(\delta',x)]}. \tag{18}$$

t_δ is determined by setting $\delta(x)=1$ for all x, where $I_c(1,x,t)$ has a maximum. At other values of x,$\delta(x)$ is fixed using this value of t_δ in Eq. (18).

Changing the variable of integration from δ to M in Eq. (18) and replacing $(\partial\delta/\partial M)_x$ with the result obtained by differentiating Eq. (15), one obtains

$$t_\delta = \int_{M(0,x,t)}^{M(\delta,x,t)} \frac{dM}{M[A(1-M) - B\ln M]R(M)} \tag{19}$$

or

$$t_\delta \equiv f(M_b, M_t). \tag{20}$$

The dependence of the integral in Eq. (19) on the exposure process is now only through the limits of integration, which are determined implicitly by Eq. (15). The subscripts b and t in Eq. (20) denote the bottom ($\delta = 1$) and top ($\delta = 0$) of the photoresist layer.

Then

$$t_x = t_d - t_\delta = \int_{x_i(\delta)}^{x_f(\delta)} \frac{dx}{R[M(\delta,x)]}. \tag{21}$$

Here $x_i(\delta)$ is the line profile calculated after a development time of t_δ and is determined from Eq. (18). The final developed image profile, given by $x_f(\delta)$, has to be determined by solving Eq. (21) for the given t_d and the t_δ obtained from Eq. (18). t_x can also be rewritten in terms of the function f of Eq. (20) by changing the variable of integration to M and recognizing that

$$\left(\frac{\partial M}{\partial x}\right)_{\delta,t} = \left(\frac{\partial h}{\partial x}\right)_t \frac{M[A(1-M) - B\ln M]}{h[A(1-h) - B\ln h]}, \tag{22}$$

where $h = h(x,t)$ is given by Eq. (16). Equation (22) is obtained by differentiating Eq. (15) with respect to x.

Substitution in Eq. (21) yields

$$t_x = \left(\frac{\partial \ln h}{\partial x}\right)_t^{-1} [A(1-h) - B\ln h]^{-1} \times \int_{M[x_i(\delta)]}^{M[x_f(\delta)]} d\ln M\{[A(1-M) - B\ln M]R(M)\}^{-1} \tag{23}$$

$$= \left(\frac{\partial \ln h}{\partial x}\right)_t^{-1} [A(1-h) - B\ln h]^{-1} f(M_f, M_i), \tag{24}$$

where M_f and M_i are used to denote the two limits of integration in Eq. (23).

V. IMAGE PROFILE EVALUATION

The procedure for evaluating the developed image profile is summarized here. Results are described in the next section.

The aerial image intensity $I_0(x)$ incident on the CEF is determined by the projection optics. $I_c(1,x,t)$ is then obtained from Eqs. (10) and (8). The lower limit of the integral in Eq. (15) is then calculated from the integral in Eq. (16). The development time t_δ required to clear the resist from the substrate in the regions where $I_c(1,x,t)$ has maxima, is obtained from Eq. (18) by setting the upper limit $\delta(x) = 1$. Equation (18) leads to the line profile at values of x, other than those corresponding to the maxima in $I_c(1,x,t)$. Therefore, the solution of Eq. (24) leads to the final image contour $x_f(\delta)$, once the model rate function $R[M]$ has been specified.

VI. RESULTS

The developed image profile is obtained for the polysilane AZ-2400 system of Hofer *et al.*[3] and the model CEL positive resist combination of Mack[13] used by him to simulate CEM-388 type materials. In both cases, the incident light intensity is calculated using the projection optics subroutine from PROLITH. The results for the two resist systems are presented separately below.

A. Polysilane–AZ2400 resist system

Since it is desired to compare the calculated profiles with experimentally determined profiles, the simulations here have been performed at the process conditions chosen by Hofer *et al.*, in their experiments. For completeness, they are listed in Table I. The development rate functions $R[M]$ employed in these calculations for the 5:1 and the 4:1 AZ2401 developers, are given explicitly by Hofer *et al.*[14] It should be noted that the exposure wavelength used by Hofer *et al.* is $\lambda = 313$ nm and that the image development using 5/1 water/AZ2401 developer requires a very long 660 s or more. The results are presented in several figures. Figure 1 presents the normalized aerial image intensity distribution. Figure 2 shows the image profile as a function of development time in a 5:1 AZ2401 developer. The effect of surface inhibition on the profile is evident. The shape of the calculated profiles agrees very well with those reported by Hofer *et al.* For comparison, image profiles obtained in the absence of the polysilane film are shown, for otherwise fixed process conditions, in the same figure. Degradation of the image by a reduction in the side wall slope and resist thinning is obvious.

TABLE I. Polysilane simulations.

Projection system	Resist parameters (AZ2400 @ 313 nm)
$\lambda = 313$ nm	$A = 0.162/\mu m$
$NA_0 = 0.167$	$B = 0.184/\mu m$
$\sigma = 0.52$	$C = 0.0128$ cm^2/mJ
Defocus = 1.87 μm	Thickness = 1.25 μm
Linewidth = 1.0 μm	
Pattern = line–space pair,	
CEL parameters (for 313 nm)	
$A_c = 8.93/\mu m$	
$B_c = 0.175/\mu m$	
$C_c = 0.0376$ cm^2/mJ	
Thickness = 0.2 μm	
Energy = 110 mJ/cm^2 (except where noted)	

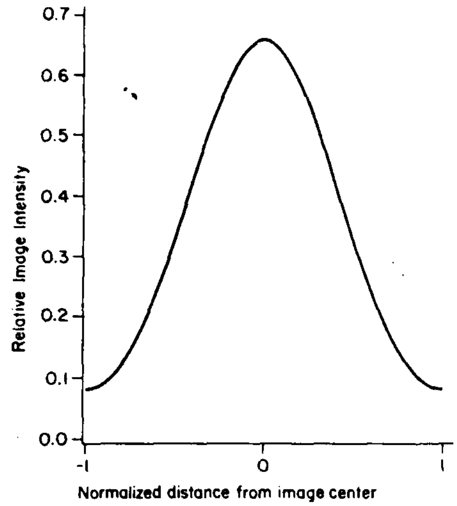

FIG. 1. Normalized aerial image intensity distribution of 1.00 μm line–space pair: $\lambda = 313$ nm, $NA_0 = 0.167$, $\sigma = 0.52$, and defocus = 1.87 μm.

Since development with the 5:1 developer takes an unduly long time, the effect of using a 4:1 developer has been studied at two exposure doses, namely, 110 and 180 mJ/cm^2. It may be noted from Fig. 3 that the 4:1 developer causes resist thinning at the lower exposure compared to the more dilute 5:1 developer, and the final image from an exposure at 180 mJ/cm^2 and development with 5:1 solution is quite superior over all the other images.

B. CEM-388–resist system

Finally, Fig. 4 shows the profiles obtained with the CEL–resist combination studied by Mack. The CEL used here is very similar to the CEM-388 manufactured by General Electric. The nominal parameters for the system are given in Table II, containing the parameters for the development rate function $R[M]$ proposed by Mack.[13] Three CEL film thicknesses are investigated: 0.2, 0.4, and 0.6 μm. Exposure energy for each thickness is adjusted to give the same nominal linewidth at the bottom of the opening after development for a fixed time of 60 s. No surface inhibition term is present in the dissolution rate function used here. The variation of the side wall angle with CEL thickness of the images in Fig. 4 is very close to that predicted by Mack from his PROLITH simulation study.

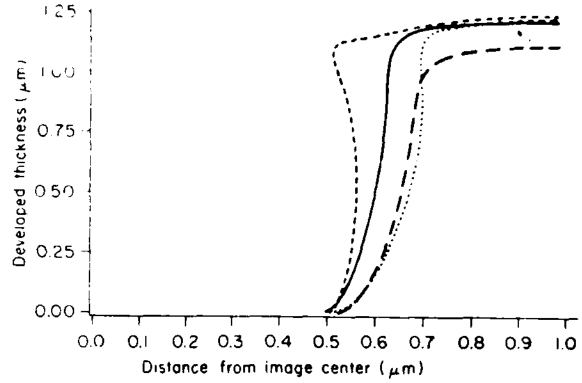

FIG. 3. Simulated profiles of 1.00 μm line–space pair in AZ2400 resist as a function of AZ2401 developer concentration for various exposure energies and development times chosen for the same nominal linewidth: — 4:1 and 180 mJ/cm^2, 60 s; --- 5:1 and 180 mJ/cm^2, 240 s; — 4:1 and 110 mJ/cm^2, 140 s; ··· 5:1 and 110 mJ/cm^2, 725 s.

VII. CONCLUSIONS

The concentration of the PAC in the underlying positive resist has been evaluated in a closed form, allowing for simultaneous bleaching of the contrast enhancing layer and the positive resist. Representing the resist development by a two stage process[10,11] the resulting image profiles have been calculated for the polysilane–AZ2400 resist system studied by Hofer *et al.*, and for a model CEM-388–resist combination investigated by Mack. Agreement with the experimen-

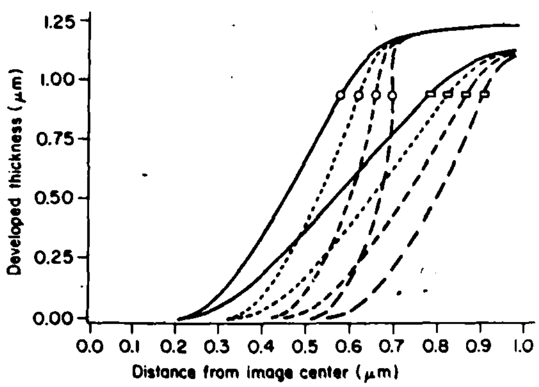

FIG. 2. Simulated profiles of 1.00 μm line–space pair in AZ2400 resist using 5:1 water/AZ2401 developer in the presence and in the absence of 0.2 μm polysilane layer at various development times: — 575, --- 625, -- 675, and ---- 725 s. O with polysilane layer and □ without polysilane layer.

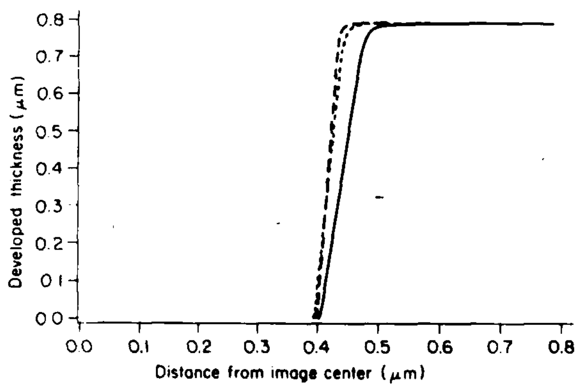

FIG. 4. Simulated profiles of 0.8 μm isolated space in model CEM–positive resist system studied by Mack — 0.2 μm and 120.0 mJ/cm^2; --- 0.4 μm and 180.0 mJ/cm^2; - - - 0.6 μm and 247.7 mJ/cm^2.

TABLE II. CEM simulations.

Projection system	Resist parameters
$\lambda = 405$ mm	$A = 0.6/\mu m$
$NA_0 = 0.28$	$B = 0.1/\mu m$
$\sigma = 0.70$	$C = 0.020\ cm^2/mJ$
No defocus	Thickness $= 0.8\ \mu m$
Linewidth $= 0.8\ \mu m$	
Pattern = space (pitch $= 4.0\ \mu m$)	
CEL parameters	**Developer conditions**
$A_c = 12.0/\mu m$	Development time = 60 s
$B_c = 0.10/\mu m$	$R_{max} = 200$ nm/s
$C_c = 0.10\ cm^2/mJ$	$R_{min} = 1$ nm/s
	$M_{TH} = 0.5$
	$n = 5$
Exposure energy	
Variable (varied for a given CEL thickness so nominal linewidth attained in 60 s development time)	

tal results of Hofer *et al.* and the calculations of Mack is very good.

ACKNOWLEDGMENTS

The authors express their gratitude to D. C. Hofer, C. Mack, and A. R. Neureuther for stimulating and useful discussions.

[a)] Supported in part by the National Science Foundation under Grant No. ECS-8611298 and by the Air Force Office of Scientific Research under research Grant No. AFOSR-87-0310.

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Commun. Math. Phys. 116, 449–474 (1988)

Communications in Mathematical Physics

Recursion Operators and Bi-Hamiltonian Structures in Multidimensions. II

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Abstract. We analyze further the algebraic properties of bi-Hamiltonian systems in two spatial and one temporal dimensions. By utilizing the Lie algebra of certain basic (starting) symmetry operators we show that these equations possess infinitely many time dependent symmetries and constants of motion. The master symmetries τ for these equations are simply derived within our formalism. Furthermore, certain new functions T_{12} are introduced, which algorithmically imply recursion operators Φ_{12}. Finally the theory presented here and in a previous paper is both motivated and verified by regarding multidimensional equations as certain singular limits of equations in one spatial dimension.

I. Introduction

This paper investigates certain algebraic aspects of exactly solvable evolution equations in 2 + 1 (i.e. in two spatial and in one temporal dimensions). It is a continuation of [1], although it can be read independently.

We consider evolution equations in the form

$$q_t = K(q), \tag{1.1}$$

where $q(x, y, t)$ is an element of a suitable space S of functions vanishing rapidly for large x, y. Let K be a differentiable map on this space and assume that it does not depend explicitly on x, y, t. If Eq. (1.1) is integrable then it belongs to some hierarchy (generated by a recursion operator Φ_{12}), hence in association with (1.1) we shall study $q_t = K^{(n)}(q)$. Fundamental in our theory is to write these equations in the form

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} \Phi_{12}^n \hat{K}_{12}^0 \cdot 1 \doteq \int_{\mathbb{R}} dy_2 \delta_{12} K_{12}^{(n)} = K_{11}^{(n)}, \tag{1.2}$$

where $\delta_{12} = \delta(y_1 - y_2)$ denotes the Dirac delta function, $q_i \doteq q(x, y_i, t)$, $i = 1, 2$,

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$K_{12}^{(n)}(q_1, q_2)$ belong to a suitably extended space $\tilde{S}$. Φ_{12}, K_{12}^0 are operator valued functions in $\tilde{S}$. If q is a matrix function then 1 is replaced by the identity matrix. Throughout this paper m and n are non-negative integers.

The following results were obtained in [1]: i) There is an algorithmic approach for obtaining the recursion operator Φ_{12} from the associated isospectral eigenvalue problem. ii) This operator is hereditary. iii) Each member of the hierarchy $(\Phi_{12}^m \hat{K}_{12}^0 \cdot 1)_{11} \doteq \int_{\mathbb{R}} dy_2 \delta_{12} \Phi_{12}^m \hat{K}_{12}^0 \cdot 1$, where $K_{12}^0 \cdot 1$ is a starting symmetry, is a symmetry of (1.2). For example the Kadomtsev Petviashvili (KP) equation and the Davey–Stewartson (DS) equation admit two such hierarchies of commuting symmetries. iv) If the hereditary operator admits a factorization in terms of two Hamiltonian operators, then hierarchies of commuting symmetries give rise to hierarchies of constants of motion in involution with respect to two different Poisson brackets. For example, the KP and the DS equations admit two such hierarchies of conserved quantities.

The above results extend the theory of [2 4] to equations in $2+1$. Novel aspects of the theory in $2+1$ include: i) The role of the Frechét derivative is now played by a certain directional derivative. If subscripts f and d denote these derivatives then there is a simple relationship between directional and total Frechét derivatives:

$$K_{12_d}[\delta_{12} F_{12}] = K_{12_f}[F] \doteq K_{12_{q_1}}[F_{11}] + K_{12_{q_2}}[F_{22}], \tag{1.3a}$$

where K_{12} is an arbitrary function in $\tilde{S}$, and $K_{12_{q_i}}$ denotes the Frechét derivative of K_{12} with respect to q_i, i.e.

$$K_{12_{q_i}}[F_{ii}] \doteq \frac{\partial}{\partial \varepsilon} K_{12}(q_i + \varepsilon F_{ii}, q_j)|_{\varepsilon=0}, \quad i, j = 1, 2, \quad i \neq j. \tag{1.3b}$$

Operators on which directional derivatives are defined are called admissible [1] (applications of the d-derivative in explicit examples can be found in Appendix A, see also Appendix C of [1]). ii) The starting symmetry K_{12}^0 can be written as $\hat{K}_{12}^0 \cdot 1$, where $\hat{K}_{12}^0$ is an admissible operator. Essential to our theory is that the operators $\hat{K}_{12}^0$, acting on suitable functions H_{12}, form a Lie algebra.

1. For the equations associated with the KP equation,

$$\Phi_{12} = D^2 + q_{12}^+ + D q_{12}^+ D^{-1} + q_{12}^- D^{-1} q_{12}^- D^{-1}, \quad q_{12}^\pm \doteq q_1 \pm q_2 + \alpha(D_1 \mp D_2), \tag{1.4}$$

where $D_i \doteq \partial/\partial y_i$. The starting operators $\hat{K}_{12}^0$ are given by

$$\hat{N}_{12} \doteq q_{12}^-, \quad \hat{M}_{12} \doteq D q_{12}^+ + q_{12}^- D^{-1} q_{12}^-, \tag{1.5}$$

and H_{12} is an arbitrary function independent of x, i.e.

$$H_{12} = H_{12}(y_1, y_2). \tag{1.6}$$

The Lie algebra of $\hat{K}_{12}^0$ is given by

$$[\hat{N}_{12} H_{12}^{(1)}, \hat{N}_{12} H_{12}^{(2)}]_d = -\hat{N}_{12} H_{12}^{(3)}, [\hat{N}_{12} H_{12}^{(1)}, \hat{M}_{12} H_{12}^{(2)}]_d = -\hat{M}_{12} H_{12}^{(3)},$$
$$[\hat{M}_{12} H_{12}^{(1)}, \hat{M}_{12} H_{12}^{(2)}]_d = -\Phi_{12} \hat{N}_{12} H_{12}^{(3)}, \tag{1.7}$$

where

$$[K_{12}^{(1)}, K_{12}^{(2)}]_d = K_{12_d}^{(1)}[K_{12}^{(2)}] - K_{12_d}^{(2)}[K_{12}^{(1)}], \tag{1.8}$$

$$H_{12}^{(3)} \doteq [H_{12}^{(1)}, H_{12}^{(2)}]_t \doteq \int_{\mathbb{R}} dy_3 (H_{13}^{(1)} H_{32}^{(2)} - H_{13}^{(2)} H_{32}^{(1)}). \tag{1.9}$$

2. For the equations associated with the DS equation

$$\Phi_{12} = \sigma(P_{12} - Q_{12}^{+} P_{12}^{-1} Q_{12}^{+}), \quad Q_{12}^{\pm} F_{12} \doteq Q_1 F_{12} \pm F_{12} Q_2,$$
$$P_{12} F_{12} \doteq F_{12_x} - J F_{12_{y_1}} - F_{12_{y_2}} J, \tag{1.10}$$

where $J = \alpha\sigma, \sigma = \operatorname{diag}(1, -1)$, Q is a 2×2 off-diagonal matrix containing the potentials $q_1(x, y), q_2(x, y)$ and Φ_{12} is defined on off-diagonal matrices. The starting operators $\hat{K}_{12}^{0}$ are given by:

$$\hat{N}_{12} \doteq Q_{12}^{-}, \quad \hat{M}_{12} \doteq Q_{12}^{-}\sigma, \tag{1.11}$$

and H_{12} is an arbitrary matrix function satisfying the following properties:

$$H_{12} \text{ diagonal matrix}, \quad P_{12} H_{12} = 0. \tag{1.12}$$

Also

$$[\hat{N}_{12} H_{12}^{(1)}, \hat{N}_{12} H_{12}^{(2)}]_d = -\hat{N}_{12} H_{12}^{(3)}, \quad [\hat{N}_{12} H_{12}^{(1)}, \hat{M}_{12} H_{12}^{(2)}]_d = -\hat{M}_{12} H_{12}^{(3)},$$
$$[\hat{M}_{12} H_{12}^{(1)}, \hat{M}_{12} H_{12}^{(2)}]_d = -\hat{N}_{12} H_{12}^{(3)}. \tag{1.13}$$

iii) The recursion operator Φ_{12} is admissible and enjoys a simple commutator operator relation with $h_{12} = h(y_1 - y_2)$:

$$[\Phi_{12}, h_{12}] = -\beta h'_{12}, \quad h'_{12} \doteq \frac{\partial h_{12}}{\partial y_1}, \tag{1.14}$$

which implies that $\delta_{12} K_{12}^{(n)} = \delta_{12} \Phi_{12}^{n} \hat{K}_{12}^{0} \cdot 1 = \sum_{l=0}^{n} \beta^l \binom{n}{l} \Phi_{12}^{n-l} \delta_{12}^{l} \hat{K}_{12}^{0} \cdot 1$, where $\delta_{12}^{l} \doteq \partial^l \delta_{12} / \partial y_1^l$.

The starting operator $\hat{K}_{12}^{0}$ is also admissible and its commutator relation with h_{12} implies that $\delta_{12} K_{12}^{(n)}$ can be written in the form

$$\delta_{12} K_{12}^{(n)} = \delta_{12} \Phi_{12}^{n} \hat{K}_{12}^{0} \cdot 1 = \sum_{l=1}^{n} b_{n,l} \Phi_{12}^{n-l} \hat{K}_{12}^{0} \cdot \delta_{12}^{l} \tag{1.15}$$

for suitable constants $b_{n,l}$.

1. For the two classes of evolution equations associated with the KP equation we have that

$$\beta = -4\alpha, [\hat{N}_{12}, h_{12}] = 0, \quad [\hat{M}_{12}, h_{12}] = -\tilde{\beta} D h'_{12}, \quad \tilde{\beta} = \beta/2, \tag{1.16}$$

and

$$b_{n,l} = \begin{cases} \beta^l \binom{n}{l}, & \text{for } \hat{K}_{12}^{0} = \hat{N}_{12} \\ \sum_{s=0}^{l} \beta^{l-s} \tilde{\beta}^s \binom{n-s}{l-s}, & \text{for } \hat{K}_{12}^{0} = \hat{M}_{12}. \end{cases} \tag{1.17}$$

2. For the two classes of evolution equations associated with the DS equation we have that

$$\beta = 2\alpha, \quad [\hat{N}_{12}, h_{12}] = [\hat{M}_{12}, h_{12}] = 0 \tag{1.18}$$

and

$$h_{n,l} = \beta^l \binom{n}{l}. \tag{1.19}$$

In [1] we assume knowledge of the underlying isospectral problem. This problem implies: a) a hereditary operator Φ_{12}; b) suitable starting operators, say $\hat{M}_{12}$ and $\hat{N}_{12}$, and functions H_{12}; c) two skew symmetric operators such that $\Phi_{12} = \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$. Furthermore, it can be shown that Φ_{12} is a strong symmetry for the starting symmetries. One then needs to: a) Find β and $h_{n,l}$ appearing in Eqs. (1.14), (1.15). b) Compute the Lie algebras of $\hat{M}_{12}, \hat{N}_{12}$ on function H_{12} (i.e. obtain equations analogous to (1.7), (1.13). c) Verify that the starting symmetries correspond to extended gradients, i.e. verify that $((\Theta_{12}^{(1)})^{-1}\hat{K}_{12}^{0}\cdot H_{12})_d$, $\hat{K}_{12}^{0} = \hat{M}_{12}$ or $\hat{N}_{12}$, is symmetric with respect to the bilinear form

$$\langle g_{12}, f_{12}\rangle \doteq \int_{\mathbb{R}^3} dx\,dy_1\,dy_2 \operatorname{trace} g_{21} f_{12}. \tag{1.20}$$

d) Verify that $\Theta_{12}^{(1)}, \Theta_{12}^{(2)}$ are compatible Hamiltonian operators.

In this paper the following results are presented. i) In Sect. 2 we investigate further the Lie algebra of the starting symmetries $\hat{K}_{12}^{0}H_{12}$. In [1] we only used a subclass of solutions of (1.6) and (1.12), given by $H_{12} = h_{12} = h(y_1 - y_2)$ and $H_{12} = h_{12}(aI + b\sigma)$, a, b, constants, respectively. This gave rise to time-independent commuting symmetries. We now choose H_{12} to be a more general solution of the above equations; this gives rise to time dependent symmetries. Time dependent symmetries for the KP have been studied in [6, 7, 18, 20]. ii) In Sect. 3, using the Lie algebra of $\hat{K}_{12}^{0}H_{12}$ and an isomorphism between Lie and Poisson brackets we prove directly that $\Phi_{12}^{n}\hat{K}_{12}^{0}H_{12}$ correspond to conserved quantities. This derivation, which capitalizes on the arbitrariness of H_{12}, has the advantage that does not use the bi-Hamiltonian factorization of Φ_{12}. In other words, for the theory developed in this paper one needs only to verify a)–c) above.

We recall that Fuchssteiner and one of the authors (ASF) introduced an alternative way for generating symmetries, the so-called master-symmetry approach. A master-symmetry is a function τ which has the property that its Lie commutator with a symmetry is also a symmetry. The τ functions for the Benjamin–Ono and the KP equations were given in [5] and [6, 7] respectively. Several authors (e.g. [8]–[12]) have noticed that master-symmetries also exist for equations in $1+1$ as well as for finite dimensional systems [13]. Let τ and T denote mastery-symmetries for equations in $2+1$ and $1+1$ respectively. If Φ is the recursion operator and $\Sigma = tK + T_0$ is the scaling symmetry of an equation in $1+1$, $q_t = K$, then $T = \Phi T_0$ is a master symmetry. However, there exists a fundamental difference between τ and T. The function $\Theta^{-1}T$ (Θ is a Hamiltonian operator) is *not* a gradient function; this can be used to constructively obtain Φ from T. But $\Theta^{-1}\tau$ is a gradient and hence the above construction of Φ from τ fails.

In Sect. 4 we show that τ is *not* the proper analogue of T. Let us consider the KP for concreteness. As it was mentioned earlier, $\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1$ generates time-

independent symmetries; it will be shown here that $\Phi_{12}^n \hat{K}_{12}^0(y_1 + y_2)^m$ generates time-dependent symmetries. It turns out that $\tau = (\Phi_{12}^2 \hat{K}_{12}^0(y_1 + y_2))_{11}$ (see Sect. IID). But $\Theta_{12}^{-1} \Phi_{12}^n \hat{K}_{12}^0 H_{12}$ is an extended gradient for all H_{12}, hence $\Theta^{-1}\tau$ is a gradient function. In Sect. 4 we show that the proper analogue of T for the KP is $T_{12} \doteq \Phi_{12}^2 \delta_{12}$ (it corresponds to $\Phi^2 \cdot 1$ for the KdV). Actually, $\Theta_{12}^{-1} T_{12}$ is *not* an extended gradient and it *can* be used to constructively obtain Φ_{12}.

In Sect. 5 we show that exactly solvable 2 + 1 dimensional equations are *exact reductions* of nonlocal evolution equations generated via nonlocal isospectral eigenvalue problems. This result both motivates the basic ideas and concepts introduced in [1] and in this paper, as well as verifies several results presented in the above papers.

II. A Lie-Algebra for Equations in 2 + 1

In developing a theory for time-dependent symmetries in 2 + 1 it is useful first to: i) characterize the commutator properties of these symmetries, ii) study the action of Φ on the Lie commutator $[a,b]_L$, where

$$[a,b]_L \doteq a_L[b] - b_L[a], \tag{2.1}$$

and a_L denotes an appropriate derivative. This derivative is linear and satisfies the Liebnitz rule. For equations in 1 + 1 one only needs $[a,b]_f$, while for equations in 2 + 1 one also needs $[a_{12}, b_{12}]_d$ (see (1.3)).

Lemma 2.1. *$\sigma^{(r)}$ is a time dependent symmetry of order r of the equation $q_t = K$,* i.e.

$$\frac{\partial \sigma^{(r)}}{\partial t} + [\sigma^{(r)}, K]_L = 0, \tag{2.2}$$

iff

$$\sigma^{(r)} = \sum_{j=0}^{r} t^j \Sigma^{(j)}, \quad \Sigma^{(j)} \doteq -\frac{1}{j}[\Sigma^{(j-1)}, K]_L, \quad j = 1, \ldots, r, \quad [K, \Sigma^{(r)}]_L = 0. \tag{2.3}$$

The above result follows from the definition of a symmetry and the assumption that $\Sigma^{(j)}$ is time independent. It implies that constructing a symmetry of order l is equivalent to finding a function $\Sigma^{(0)}$ with the property that its $(l+1)^{st}$ commutator with K is zero.

The action of a hereditary operator Φ on a Lie commutator is given by:

Theorem 2.1. *Let*

$$S \doteq \Phi_L[K] + [\Phi, K_L]. \tag{2.4}$$

Then

a$_1$)
$$\Phi^n[K_1, K_2]_L = [K_1, \Phi^n K_2]_L + \left(\sum_{r=1}^{n} \Phi^{n-r} S_1 \Phi^{r-1}\right) K_2. \tag{2.5}$$

If Φ is hereditary, i.e. *if*

$$\Phi_L[\Phi v]w - \Phi\Phi_L[v]w \text{ is symmetric with respect to } v, w, \tag{2.6}$$

then the following are true:

a$_2$) $\Phi_L[\Phi^n K] + [\Phi, (\Phi^n K)_L] = \Phi^n S$, (2.7)

a$_3$) $\Phi^{n+m}[K_1, K_2]_L = [\Phi^n K_1, \Phi^m K_2]_L$

$$+\Phi^n\left(\sum_{r=1}^{m} \Phi^{m-r} S_1 \Phi^{r-1}\right) K_2 - \Phi^m\left(\sum_{r=1}^{n} \Phi^{n-r} S_2 \Phi^{r-1}\right) K_1. \tag{2.8}$$

(*m, n are non-negative integers*).

Proof. To prove (2.5) use induction: $(2.5)_0$ is an identity. Applying Φ on $(2.5)_n$ we obtain

$$\Phi^{n+1}[K_1, K_2]_L = \Phi[K_1, \Phi^n K_2]_L + \Phi\left(\sum_{r=1}^{n} \Phi^{n-r} S_1 \Phi^{r-1}\right) K_2.$$

Equation $(2.5)_{n+1}$ follows from the above and the following identity

$$\Phi[K_1, M]_L = [K_1, \Phi M]_L + S_1 M.$$

Equation (2.7) also follows from induction. To prove (2.8) first note that (2.5) implies

$$\Phi^m[K_1, K_2]_L - \left(\sum_{r=1}^{m} \Phi^{m-r} S_1 \Phi^{r-1}\right) K_2 = [K_1, \Phi^m K_2]_L. \tag{2.9}$$

Equation (2.5) also implies

$$\Phi^n[K_1, \tilde{K}_2]_L = [\Phi^n K_1, \tilde{K}_2]_L - \left(\sum_{r=1}^{n} \Phi^{n-r} \tilde{S}_2 \Phi^{r-1}\right) K_1.$$

Let $\tilde{K}_2 = \Phi^m K_2$, then (2.6) implies $\tilde{S}_2 = \Phi^m S_2$, and the above equation becomes

$$\Phi^n[K_1, \Phi^m K_2]_L = [\Phi^n K_1, \Phi^m K_2]_L - \left(\sum_{r=1}^{n} \Phi^{n-r} \Phi^m S_2 \Phi^{r-1}\right) K_1.$$

Applying Φ^n on (2.9) and using the above we obtain (2.8).

Corollary 2.1. *Let the hereditary operator Φ be a strong symmetry for both K_1 and K_2*, i.e. $S_1 = S_2 = 0$. *Then*

$$\Phi^{n+m}[K_1, K_2]_L = [\Phi^n K_1, \Phi^m K_2]_L. \tag{2.10}$$

In the rest of this section we characterize extended symmetries σ_{12}. The following theorem, proven in [1], maps extended symmetries σ_{12} to symmetries σ_{11}.

Theorem 2.2. *Assume that the commutator of Φ_{12} with h_{12} is given by (1.14) and that the starting operator $\hat{K}^0_{12}$ are such that (1.15) is valid. If σ_{12} is an extended symmetry of (1.2)*, i.e. *if*

$$\frac{\partial \sigma_{12}}{\partial t} + [\sigma_{12}, \delta_{12} \Phi_{12}^n \hat{K}^0_{12} \cdot 1]_d = 0, \tag{2.11}$$

then σ_{11} is a symmetry of (1.2), i.e.

$$\frac{\partial \sigma_{11}}{\partial t} + [\sigma_{11}, K_{11}^{(n)}]_f = 0. \tag{2.12}$$

In the above

$$[\sigma_{11}, K_{11}^{(n)}]_f = \sigma_{11_{q_1}}[K_{11}^{(n)}] - K_{11_{q_1}}^{(n)}[\sigma_{11}], \tag{2.13}$$

and

$$[\sigma_{12}, \delta_{12}\Phi_{12}^n \hat{K}_{12}^0 \cdot 1]_d = \sum_{l=0}^{n} b_{n,l}[\sigma_{12}, \Phi_{12}^{n-l}\hat{K}_{12}^0 \delta_{12}^l]_d. \tag{2.14}$$

It is necessary to rewrite $\delta_{12}\Phi_{12}^n\hat{K}_{12}^0 \cdot 1$ in the form appearing in (2.14) since the directional derivative is defined only for functions of the form $\hat{L}_{12}H_{12}$, where $\hat{L}_{12}$ is an admissible operator.

Using Lemma 2.1, Corollary 2.1 and the Lie algebra of $\hat{K}_{12}^0 H_{12}$ (with appropriate H_{12}) we obtain extended symmetries, which then via Theorem 2.2 give rise to symmetries.

Proposition 2.1. *Assume that the hereditary operator Φ_{12} is a strong symmetry for the admissible starting operators $\hat{M}_{12}, \hat{N}_{12}$, and that (1.14), (1.15) hold. Further assume that $\hat{M}_{12}, \hat{N}_{12}$ form a Lie algebra (analogous to (1.7), (1.13)). Consider the following hierarchies*

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12}\Phi_{12}^n \hat{N}_{12} \cdot 1 = \int_{\mathbb{R}} dy_2 \delta_{12} N_{12}^{(n)} = N_{11}^{(n)}, \tag{2.15a}$$

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12}\Phi_{12}^n \hat{M}_{12} \cdot 1 = \int_{\mathbb{R}} dy_2 \delta_{12} M_{12}^{(n)} = M_{11}^{(n)}. \tag{2.15b}$$

Then:

a) $(\Phi_{12}^m \hat{M}_{12} \cdot 1)_{11}, (\Phi_{12}^m \hat{N}_{12} \cdot 1)_{11}$, *are symmetries of Eqs. (2.15).*

b) *Appropriate linear combinations of* $\{\Phi_{12}^m \hat{M}_{12} H_{12}^{(r)}\}_{11}, \{\Phi_{12}^m \hat{N}_{12} H_{12}^{(r)}\}_{11}$ *for suitable functions $H_{12}^{(r)}$ generate time dependent symmetries for Eqs. (2.15).*

Rather than proving the above proposition in general, we use for concreteness, the Lie algebra (1.6) to sketch how the above results can be derived. Details are given in II.A, II.B. Let

$$\hat{N}_{12}^{(n)} \doteq \Phi^n \hat{N}_{12}, \quad \hat{M}_{12}^{(n)} \doteq \Phi^n \hat{M}_{12}. \tag{2.16}$$

Then, using Corollary 2.1, Eqs. (1.7) imply

$$\begin{aligned}
[\hat{N}_{12}^{(m)} H_{12}^{(1)}, \hat{N}_{12}^{(n-l)} H_{12}^{(2)}]_d &= -\hat{N}^{(m+n-l)} H_{12}^{(3)}, \\
[\hat{N}_{12}^{(m)} H_{12}^{(1)}, \hat{M}_{12}^{(n-l)} H_{12}^{(2)}]_d &= -\hat{M}_{12}^{(m+n-l)} H_{12}^{(3)}, \\
[\hat{M}_{12}^{(m)} H_{12}^{(1)}, \hat{N}_{12}^{(n-l)} H_{12}^{(2)}]_d &= -\hat{M}_{12}^{(m+n-l)} H_{12}^{(3)}, \\
[\hat{M}_{12}^{(m)} H_{12}^{(1)}, \hat{M}_{12}^{(n-l)} H_{12}^{(2)}]_d &= -\hat{N}_{12}^{(m+n+1-l)} H_{12}^{(3)}.
\end{aligned} \tag{2.17}$$

Part a) of the proposition is a direct consequence of Eqs. (2.17) and (2.14). For example

$$[\hat{N}_{12}^{(m)} \cdot 1, \delta_{12}\hat{N}_{12}^{(n)} \cdot 1]_d = -\sum_{l=0}^{n} b_{n,l}\hat{N}^{(m+n-l)} \cdot \tilde{H}_{12}^{(l)} = 0,$$

since $\tilde{H}_{12}^{(l)} = [1, \delta_{12}^l]_l = 0$; thus $\hat{N}_{12}^{(m)} \cdot 1$ are extended symmetries of (2.15a).

Consider part b) of Proposition 2.1. Let us first consider symmetries of order

one it t. Then

$$\hat{N}_{12}^{(m)}(y_1+y_2) - t2\beta\binom{n}{1}\hat{N}_{12}^{(m+n-1)}\cdot 1,$$

$$\hat{M}_{12}^{(m)}(y_1+y_2) - t2\beta\binom{n}{1}\hat{M}_{12}^{(m+n-1)}\cdot 1 \tag{2.18}$$

are first order time dependent extended symmetries of (2.15a). Similarly

$$\hat{N}_{12}^{(m)}\cdot(y_1+y_2) - t2b_{n,1}\hat{M}_{12}^{(n+m-1)}\cdot 1, \tag{2.19a}$$

$$\hat{M}_{12}^{(m)}\cdot(y_1+y_2) - t2b_{n,1}\hat{N}^{(m+n)}\cdot 1, \tag{2.19b}$$

are extended symmetries of (2.15b) with $b_{n,l} = (-4\alpha)\sum_{s=0}^{l} 2^{-s}\binom{n-s}{l-s}$.

To derive the above we use Lemma 2.1 and Eqs. (2.17). For example, to derive (2.18) we look for a function $\Sigma_{12}^{(0)}$ such that its commutator with $\delta_{12}\hat{N}_{12}^{(n)}\cdot 1$, commutes with $\delta_{12}\hat{N}_{12}^{(n)}\cdot 1$. Clearly $\Sigma_{12}^{(0)} = \hat{N}_{12}^{(m)}(y_1+y_2)$ or $\hat{M}_{12}^{(m)}(y_1+y_2)$. For, (2.17a) implies

$$[\hat{N}_{12}^{(m)}(y_1+y_2), \delta_{12}\hat{N}_{12}^{(n)}\cdot 1]_d = 2\beta\binom{n}{1}\hat{N}^{(m+n-1)}\cdot 1,$$

since $\tilde{H}_{12}^{(l)} = [y_1+y_2, \delta_{12}^l]_l = -2\delta_{1,l}$, where $\delta_{1,l}=0$ if $l\neq 1$ or 1 if $l=1$.

In a similar manner

$$\hat{N}_{12}^{(m)}(y_1+y_2)^2 - t4\beta\binom{n}{1}\hat{N}_{12}^{(m+n-1)}(y_1+y_2) - t^2 4\beta^2\binom{n}{1}^2\hat{N}_{12}^{(m+2n-2)}\cdot 1,$$

$$\hat{M}_{12}^{(m)}(y_1+y_2)^2 - t4\beta\binom{n}{1}\hat{M}_{12}^{(m+n-1)}(y_1+y_2) + t^2 4\beta^2\binom{n}{1}^2\hat{M}_{12}^{(m+2n-2)}\cdot 1 \tag{2.20}$$

are second order time dependent extended symmetries of (2.15b). Similarly

$$\hat{N}^{(m)}\cdot(y_1+y_2)^2 - t4b_{n,1}\hat{M}^{(m+n-1)}\cdot(y_1+y_2) + t^2 4b_{n,1}^2\hat{N}^{(m+2n-1)}\cdot 1, \tag{2.21a}$$

$$\hat{M}^{(m)}\cdot(y_1+y_2)^2 - t4b_{n,1}\hat{N}^{(m+n)}\cdot(y_1+y_2) + t^2 4b_{n,1}^2\hat{M}^{(m+2n-1)}\cdot 1,$$

$$b_{n,1} = (-4\alpha)(n+\tfrac{1}{2}), \tag{2.21b}$$

are extended symmetries of (2.15b). Indeed

$$[\hat{N}_{12}^{(m)}(y_1+y_2)^2, \delta_{12}\hat{N}_{12}^{(n)}\cdot 1]_d = 4\beta\binom{n}{1}\hat{N}_{12}^{(m+n-1)}(y_1+y_2),$$

since, $[(y_1+y_2)^2, \delta_{12}^l] = -4(y_1+y_2)\delta_{1,l}$. Also

$$[\hat{N}_{12}^{(m+n-1)}(y_1+y_2), \delta_{12}\hat{N}_{12}^{(n)}\cdot 1] = 2\beta\binom{n}{1}\hat{N}_{12}^{(m+2n-2)}.$$

The extension of the above results to any order in time is straightforward: To generate $\sigma_{12}^{(r)}$ consider $\Sigma_{12}^{(0)} = \hat{N}_{12}^{(m)}(y_1+y_2)^r$ or $\hat{M}_{12}^{(m)}(y_1+y_2)^r$. The commutator of $(y_1+y_2)^r$ with δ_{12}^l produces $(y_1+y_2)^{r-l}$. Thus the r^{th} commutator of $(y_1+y_2)^r$

with δ^{1}_{12} produces 1 which commutes with $\delta^{(l)}_{12}$; hence Lemma 2.1 guarantees the existence of an r^{th} order symmetry.

II.A. Time Dependent Symmetries for the Equations Associated with the KP *Equation.* Following the construction and the argument sketched above, extended symmetries of order r in time

$$\sigma^{(r)}_{12} = \sum_{j=0}^{r} t^j \Sigma^{(j)}_{12} \tag{2.22}$$

are generated through Proposition 2.1, starting with $\Sigma^{(0)}_{12} = \hat{N}^{(m)}_{12}\cdot H^{(r)}_{12}$ or $\hat{M}^{(m)}_{12}\cdot H^{(r)}_{12}$, where $H^{(r)}_{12}$, is defined by

$$H^{(r)}_{12} \doteq (y_1 + y_2)^r; \tag{2.23}$$

more generally, any homogeneous polynomial of degree r in y_1 and y_2 could be used as well (note H^r_{12} solves (1.6)). Using

$$[H^{(r)}_{12}, \delta^s_{12}] = -(1-(-1)^s)\theta(r-s)\frac{r!}{(r-s)!}H^{(r-s)}_{12}, \tag{2.24}$$

$$\theta(a) = \begin{cases} 1, & a \geqq 0, \\ 0, & a < 0, \end{cases} \tag{2.25}$$

we can show that

i) The class of evolution equations (2.15a) with $\hat{N}_{12} = q^{-}_{12}$ admits t-dependent symmetries of order r given by

$$\Sigma^{(0)}_{12} = \hat{N}^{(m)}_{12}\cdot H^{(r)}_{12}, \tag{2.26a}$$

$$\Sigma^{(j)}_{12} = \Sigma\, v(r,j,s)\hat{N}_{12}^{\left(m+jn-\sum_{l=1}^{j}2s_l+1\right)}\cdot H_{12}^{\left(r-\sum_{l=1}^{j}2s_l+1\right)}, \tag{2.26b}$$

and by

$$\Sigma^{(0)}_{12} = \hat{M}^{(m)}_{12}\cdot H^{(r)}_{12}, \tag{2.27a}$$

$$\Sigma^{(j)}_{12} = \Sigma\, v(r,j,s)\hat{M}^{\left(m+jn-\sum_{l=1}^{j}2s_l+1\right)}\cdot H_{12}^{\left(r-\sum_{l=1}^{j}2s_l+1\right)}, \tag{2.27b}$$

where $j \geqq 1$, the summation Σ is from $s_1, s_2, \ldots, s_j$ zero to P_n and $P_n = (n-1)/2$ if n is odd and $(n-2)/2$ if n is even. Also

$$v(r,j,s) \doteq \frac{(-2)^j}{j!}\left(\prod_{\mu=1}^{j}\theta\left(r-\sum_{l=1}^{\mu}2s+1\right)\right)\left(\prod_{l=1}^{j}b_{n,2s_l+1}\right)\frac{r!}{\left(r-\sum_{l=1}^{j}2s_l+1\right)!}, \tag{2.28}$$

and $b_{n,l} = (-4\alpha)^l\binom{n}{l}$.

ii) The KP class (2.15b) with $\hat{M}_{12} = Dq^{+}_{12} = q_{12}D^{-1}q_{12}$ admits t-dependent symmetries of order r given by

$$\Sigma^{(0)}_{12} = \hat{N}^{(m)}_{12}\cdot H^{(r)}_{12}, \tag{2.29a}$$

$$\Sigma_{12}^{(2j)} = \Sigma v(r, 2j, s)\hat{N}_{12}^{\left(m+2jn+j-\sum_{l=1}^{2j} 2s_l+1\right)} \cdot H_{12}^{\left(r-\sum_{l=1}^{2j} 2s_l+1\right)}, \tag{2.29b}$$

$$\Sigma_{12}^{(2j-1)} = \Sigma v(r, 2j-1, s)\hat{M}_{12}^{\left(m+(2j-1)n+j-1-\sum_{l=1}^{2j-1} 2s_l+1\right)} \cdot H_{12}^{\left(r-\sum_{l=1}^{2j-1} 2s_l+1\right)}, \tag{2.29c}$$

and by

$$\Sigma_{12}^{(0)} = \hat{M}_{12}^{(m)} \cdot H_{12}^{(r)}, \tag{2.30a}$$

$$\Sigma_{12}^{(2j)} = \Sigma v(r,2j, s)\hat{M}_{12}^{\left(m+2jn+j-\sum_{l=1}^{2j} 2s_l+1\right)} \cdot H_{12}^{\left(r-\sum_{l=1}^{2j} 2s_l+1\right)}, \tag{2.30b}$$

$$\Sigma_{12}^{(2j-1)} = \Sigma v(r, 2j-1, s)\hat{N}_{12}^{\left(m+(2j-1)n+j-\sum_{l=1}^{2j-1} 2s_l+1\right)} \cdot H_{12}^{\left(r-\sum_{l=1}^{2j-1} 2s_l+1\right)}, \tag{2.30c}$$

with $j \geqq 1$ and $b_{n,l} = \sum_{s=0}^{l} \beta^{l-s}\tilde{\beta}^s \binom{n-s}{l-s} = (-4\alpha)^l \sum_{s=0}^{l} 2^{-s} \binom{n-s}{l-s}$.

II.B. Time Dependent Symmetries for the Equations Associated with the Davey–Stewartson Equation. The construction of t-dependent symmetries for the equations associated with the DS equation is similar. Extended symmetries of order r in time are generated through Lemma 2.1, starting with $\Sigma_{12}^{(0)} = \hat{N}^{(m)}H_{12}^{(r)}$ or $\hat{M}_{12}^{(m)}H_{12}^{(r)}$, where $H_{12}^{(r)}$ is defined by,

$$H_{12}^{(r)} \doteq \operatorname{diag}(\xi_{+_{12}}^r, \xi_{-_{12}}^r), \quad \xi_{\pm_{12}} \doteq y_1 + y_2 \pm 2\alpha x. \tag{2.31}$$

$H_{12}^{(r)}$ satisfies the same formula (2.24), obviously replacing $[H_{12}^{(r)}, \delta_{12}^s]_l$ by $[H_{12}^{(r)}, \delta_{12}^s I]_l$. Then, using Corollary 2.1 and Eqs. (1.13), one can show that

i) The class of evolution equations (2.15a) with $\hat{N}_{12} = Q_{12}^-$ admits t-dependent symmetries of order r given by Eqs. (2.26) and (2.27), where $b_{n,l} = \beta^l \binom{n}{l} = (2\alpha)^l \binom{n}{l}$ and $j \geqq 1$.

ii) The class of evolution equations (2.15b) with $\hat{M}_{12} = Q_{12}^- \sigma$ admits t-dependent symmetries of order r given by Eqs. (2.29–30), replacing: $\hat{N}^{(\cdot)} \to \hat{N}^{(\cdot-j)}$ in Eq. (2.29b), $\hat{M}^{(\cdot)} \to \hat{M}^{(\cdot-j+1)}$ in Eq. (2.29c), $\hat{M}^{(\cdot)} \to \hat{M}^{(\cdot-j)}$ in Eq. (2.30b), $\hat{N}^{(\cdot)} \to \hat{N}^{(\cdot-j)}$ in Eq. (2.30c) and using $b_{n,l} = (2\alpha)^l \binom{n}{l}$.

II.C. Connection with Known Results. Before the discovery [14] of the recursion operator of the KP equation, a different approach, the so-called master-symmetries approach, was used to generate an infinite sequence of commuting symmetries [6], as well as t-dependent symmetries [7–11], of the KP equation (see also [18, 19]).

The existence of a hereditary operator in 2 + 1 dimensions, together with the Lie algebra of the starting symmetries allows a simple and elegant characterization of the 2 + 1 dimensional (gradient) master-symmetries introduced in the above papers. Here we briefly consider the KP example.

In Proposition 2.1 and in Sect. II.B. we have shown that the functions

$$\tau_{12}^{(m,r)} \doteq \Phi_{12}^{m} K_{12}^{0} H_{12}^{(r)}, \tag{2.32}$$

(where $H_{12}^{(r)}$ is defined in (2.23), but it could be any homogeneous polynomial of degree r in y_1, y_2, and $\hat{K}_{12}^{0}$ is $\hat{N}_{12}$ or $\hat{M}_{12}$) have the property that their $(r+1)^{st}$ commutator with $\delta_{12} K_{12}^{(n)}$ is zero, namely

$$\underbrace{[\cdots}_{r+1\text{ times}} [\tau_{12}^{(m,r)}, \delta_{12} K_{12}^{(n)}]_d \underbrace{\cdots]_d}_{r+1\text{ times}} = 0. \tag{2.33}$$

Then Theorem 4.1 of [1] implies that

$$\underbrace{[\cdots}_{r+1\text{ times}} [\tau_{11}^{(m,r)}, K_{11}^{(n)}]_f \underbrace{\cdots]_f}_{r+1\text{ times}} = 0, \tag{2.34}$$

namely $\tau_{11}^{(m,r)}$ are the so-called master-symmetries of degree r of KP [11]. Equation (2.33) essentially follows from the fact that a single commutator of $\tau_{12}^{(m,r)}$ with $\delta_{12} K_{12}^{(n)}$ generates a linear combination of lower degree master-symmetries; in fact, choosing for concreteness $\tau_{12}^{(m,r)} = \Phi_{12}^{m} \hat{N}_{12}(y_1 + y_2)^r$ and $K_{12}^{(n)} = M_{12}^{(n)}$, we have

$$\begin{aligned}[\tau_{12}^{(m,r)}, \delta_{12} M_{12}^{(n)}]_d &= -\sum_{l=0}^{n} b_{n,l} \hat{M}_{12}^{(m+n)} [(y_1 + y_2)^r, \delta_{12}^{l}] \\ &= \sum_{l=1}^{n} \theta(r-l) \frac{r!}{(r-l)!} b_{n,l} \tau_{12}^{(m+n,r-l)},\end{aligned} \tag{2.35}$$

which implies

$$[\tau_{11}^{(m,r)}, M_{11}^{(n)}]_f = \sum_{l=1}^{n} \theta(r-l) \frac{r!}{(r-l)!} b_{n,l} \tau_{11}^{(m+n,r-l)}. \tag{2.36}$$

For $r = 1$ Eq. (2.36) becomes

$$[\tau_{11}^{(m,1)}, M_{11}^{(n)}]_f = b_{n,1} M_{11}^{(m+n)}; \tag{2.37}$$

master-symmetries of degree 1 generate equations which belong to the given hierarchy.

III. Lie and Poisson Brackets for Equations in 2 + 1

In this section we first derive an isomorphism between Lie and Poisson brackets. Then, using this isomorphism and the Lie algebra of the operators $\hat{K}_{12}^{0}$, we prove that $\Theta_{12}^{-1} \hat{K}_{12}^{0} H_{12}$ are extended gradients. This implies that all extended symmetries of the previous section give rise to conserved quantities.

Theorem 3.1. *Let $[a, b]_L = a_L[b] - b_L[a]$ be a Lie commutator and $\langle f, g \rangle$ be an appropriate symmetric bi-linear form. Let* grad *I be the gradient of a functional I, defined by $I_L[v] = \langle \text{grad}\, I, v \rangle$; then γ is a gradient function* iff *$\gamma_L = \gamma_L^{*}$, where M^{*} denotes the adjoint of the operator M with respect to the above bi-linear form*, i.e. *$\langle M^{*} f, g \rangle = \langle f, M g \rangle$. Then if the operator Θ is a Hamiltonian operator*, i.e. *if*

$$\Theta^{*} = -\Theta, \quad \langle a, \Theta_L[\Theta b] c \rangle + \text{cyclic permut} = 0, \tag{3.1}$$

it follows that

$$[\Theta f, \Theta g]_L = \Theta \operatorname{grad}\langle f, \Theta g\rangle + \Theta\{(f_L - f_L^*)[\Theta g] - (g_L - g_L^*)[\Theta f]\}. \quad (3.2)$$

Proof.

$$\begin{aligned}\operatorname{grad}\langle f, \Theta g\rangle[v] &= \langle f_L[v], \Theta g\rangle + \langle f, \Theta_L[v]g\rangle + \langle f, \Theta g_L[v]\rangle \\ &= \langle f_L^*[\Theta g] - g_L^*[\Theta f]\rangle = \langle f_L^*[\Theta g] + M_g^* f - g_L^*[\Theta g], v\rangle,\end{aligned}$$

where $\langle f, \Theta_L[v]g\rangle = \langle f, M_g[v]\rangle$ and M_g denotes a linear operator depending on g. Hence

$$\begin{aligned}[\Theta f, \Theta g]_L - \Theta \operatorname{grad}\langle f, \Theta g\rangle &= \Theta_L[\Theta g]f + \Theta f_L[\Theta g] - \Theta_L[\Theta f]g - \Theta g_L[\Theta f] \\ &\quad - \Theta f_L^*[\Theta g] + \Theta g_L^*[\Theta g] - \Theta M_L^* f \\ &= \Theta_L[\Theta g]f - \Theta_L[\Theta f]g - \Theta M_g^* f + \Theta\{(f_L - f_L^*)[\Theta g] - (g_L - g_L^*)[\Theta f]\}.\end{aligned}$$

But the sum of the first three terms of the above equals zero because of (3.1). Hence (3.2) follows.

In the above a_L denotes an appropriate directional derivative. For equations in $1+1$:

$$[a, b]_L = [a, b]_f, \quad \langle f, g\rangle = \int_{\mathbb{R}} dx \operatorname{trace} gf. \quad (3.3)$$

For equations in $2+1$,

$$[a_{12}, b_{12}]_L = [a_{12}, b_{12}]_d, \quad \langle f_{11}, g_{11}\rangle = \int_{\mathbb{R}^2} dx dy \operatorname{trace} g_{11} f_{11},$$

$$\langle f_{12}, g_{12}\rangle = \int_{\mathbb{R}^3} dx dy_1 dy_2 \operatorname{trace} g_{21} f_{12} \quad (3.4)$$

(if f and g are scalars, then delete trace), where $[\ ,\ ]_f$, $[\ ,\ ]_d$ are defined in (2.13), (2.4). Furthermore the following double representation of the functional I

$$I = \int_{\mathbb{R}^2} dx dy_1 \operatorname{trace} \rho_{11} = \int_{\mathbb{R}^3} dx dy_1 dy_2 \operatorname{trace} \rho_{12} \quad (3.5)$$

allows us to define the extended gradient $\operatorname{grad}_{12} I$ and the gradient $\operatorname{grad} I$ of the functional I by

$$I_d[v_{12}] = \int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} \operatorname{trace} \rho_{12}[v_{12}] \doteqdot \langle \operatorname{grad}_{12} I, v_{12}\rangle, \quad (3.6a)$$

$$I_f[v_{11}] = \int_{\mathbb{R}^2} dx dy_1 \operatorname{trace} \rho_{11_f}[v_{11}] \doteqdot \langle \operatorname{grad} I, v_{11}\rangle. \quad (3.6b)$$

The following theorem, proven in [1], maps extended gradients γ_{12} to gradients γ_{11}:

Theorem 3.2.

a) *γ_{12} and γ_{11} are extended gradients and gradients respectively* iff $\gamma_{12_d}^* = \gamma_{12_d}$ *and* $\gamma_{11_f}^* = \gamma_{11_f}$, *with respect to the bilinear forms (3.4c) and (3.4b) respectively.*

b) *If γ_{12} is an extended gradient, then γ_{11} is a gradient corresponding to the same potential, namely if $\gamma_{12} = \operatorname{grad}_{12} I$, then $\gamma_{11} = \operatorname{grad} I$.*

Proposition 3.1. *Assume that the hereditary operator Φ_{12} is a strong symmetry for the starting symmetries $\hat{M}_{12}H_{12}$ and $\hat{N}_{12}H_{12}$. Further assume that $\hat{M}_{12}, \hat{N}_{12}$ form a Lie algebra (analogous to (1.7) and (1.13)) and that Θ_{12} is a Hamiltonian operator whose inverse exists. Then*

$$\Theta_{12}^{-1}\Phi_{12}^{m}\hat{K}_{12}^{0}H_{12}, \quad \hat{K}_{12}^{0} = \hat{M}_{12} \quad or \quad \hat{N}_{12} \tag{3.7}$$

are extended gradients, proved that $\Theta_{12}^{-1}\hat{K}_{12}^{0}H_{12}$ are extended gradients.

Proof. For concreteness we proof the above proposition for the recursion operator and starting symmetries associated with the two dimensional Schrödinger and 2×2 AKNS problems.

III.A. Conserved Quantities for Equations Related to KP *Equation*

Corollary 3.1. *Let*

$$\begin{aligned} &\hat{N}_{12} \doteq q_{12}^{-}, \quad \hat{M}_{12} \doteq Dq_{12}^{+} + q_{12}^{-}D^{-1}q_{12}^{-}, \quad H_{12} \doteq H(y_1, y_2), \\ &\hat{M}_{12}^{(n)} \doteq \Phi_{12}^{n}\hat{M}_{12}, \quad \hat{N}_{12}^{(n)} = \Phi_{12}^{n}\hat{N}_{12}, \quad \Theta_{12} = D, \end{aligned} \tag{3.8}$$

where Φ_{12} is the recursion operator associated with the KP *and is defined by (1.4). Then*

$$\begin{aligned} D^{-1}\hat{M}^{(n+1)}H_{12}^{(3)} &= \operatorname{grad}\langle \hat{M}^{(n)}H_{12}^{(1)}, D^{-1}\hat{N}^{(1)}H_{12}^{(2)}\rangle, \\ D^{-1}\hat{N}^{(n+1)}H_{12}^{(3)} &= \operatorname{grad}\langle \hat{M}^{(n)}H_{12}^{(1)}, D^{-1}\hat{M}H_{12}^{(2)}\rangle. \end{aligned} \tag{3.9}$$

Proof. We first note that the assumptions of Proposition 3.1 are fulfilled. Namely Φ_{12} is hereditary and is a strong symmetry of $\hat{M}_{12}H_{12}, \hat{N}H_{12}$, (see Lemma 4.2 and Appendix C.1a of [1]). The operator D^{-1} is obviously a Hamiltonian operator. Furthermore, $D^{-1}\hat{M}_{12}H_{12}$ is an extended gradient (see Appendix A). Since $D^{-1}\hat{M}_{12}H_{12}$ is an extended gradient, Theorem 3.1 and (1.7c) imply that $D^{-1}\hat{N}_{12}^{(1)}H_{12}$ is an extended gradient. Then Theorem 3.1 and $[\hat{M}^{(n)}H_{12}^{(1)}, \hat{N}^{(1)}H_{12}^{(2)}]_d = -\hat{M}^{(n+1)}H_{12}^{(3)}$ imply by induction (3.9a). Finally Theorem 3.1 and $[\hat{M}^{(n)}H_{12}^{(1)}, \hat{M}H_{12}^{(2)}]_d = -\hat{N}^{(n+1)}H_{12}^{(3)}$ imply by induction (3.9b).

A consequence of the above result is that all symmetries derived in Sect. II.B. give rise to conserved quantities. For example, the following t-dependent extended symmetries (see (2.19b) and (2.21a))

$$\begin{aligned} \sigma_{12}^{(1)} &= \hat{M}_{12}^{(m)}(y_1 + y_2) + t12\alpha\hat{N}_{12}^{(m+1)} \cdot 1, \\ \sigma_{12}^{(2)} &= \hat{N}_{12}^{(m)}(y_1 + y_2)^2 + t24\alpha\hat{M}_{12}^{(m)} \cdot (y_1 + y_2) + t^2 144\alpha^2\hat{N}_{12}^{(m+1)} \cdot 1, \end{aligned}$$

of the KP equation $q_{1_t} = M_{11}^{(1)} = 2(q_{1_{xxx}} + 6q_1q_{1_x} + 3\alpha^2D^{-1}q_{1_{y_1y_1}})$ correspond to extended gradient functions $D^{-1}\sigma_{12}^{(1)}$ and $D^{-1}\sigma_{12}^{(2)}$; then they give rise to the following t-dependent conserved quantities (see Eqs. (4.15))

$$I^{(1)} = \int_{\mathbb{R}^2} dx\,dy_1\left(\frac{1}{2(2m+3)}(D^{-1}\hat{M}_{12}^{(m+1)}(y_1 + y_2))_{11} + \frac{3\alpha t}{m+2}(D^{-1}\hat{N}_{12}^{(m+2)} \cdot 1)_{11}\right),$$

$$I^{(2)} = \int_{\mathbb{R}^2} dx\,dy_1\left(\frac{1}{4(m+1)}(D^{-1}\hat{N}^{(m+1)}(y_1 + y_2)^2)_{11}\right.$$

$$+\frac{t12\alpha}{2m+3}(D^{-1}\hat{M}_{12}^{(m+1)}(y_1+y_2))_{11}+\frac{t^2 36\alpha^2}{m+2}(D^{-1}\hat{N}_{12}^{(m+2)}\cdot 1)_{11}\Big).$$

III.B. Conserved Quantities for Equations Related to DS Equation

Corollary 3.2. *Let*

$$\hat{M}_{12} \doteq Q_{12}\sigma, \quad \hat{N}_{12} \doteq Q_{12},$$

H_{12} diagonal and such that

$$P_{12}H_{12}=0, \quad \hat{M}_{12}^{(n)} \doteq \Phi_{12}^n \hat{M}_{12}, \quad \hat{N}_{12}^{(n)} = \Phi_{12}^n \hat{N}_{12}, \quad \Theta_{12}=\sigma, \tag{3.10}$$

where Φ_{12} is the recursion operator associated with the DS equation and is defined by (1.9). Then

$$\begin{aligned} \sigma\hat{M}^{(n+1)}H_{12}^{(3)} &= \operatorname{grad}\langle \hat{M}^{(n)}H_{12}^{(1)}, \sigma\hat{N}^{(1)}H_{12}^{(2)}\rangle, \\ \sigma\hat{N}^{(n)}H_{12}^{(3)} &= \operatorname{grad}\langle \hat{M}^{(n)}H_{12}^{(1)}, \sigma\hat{M}H_{12}^{(1)}\rangle. \end{aligned} \tag{3.11}$$

Proof. The assumptions of Proposition 3.1 are again fulfilled (see Lemma 4.2 and Appendix C.2a of [1]). The operator σ is obviously Hamiltonian in a space of off-diagonal matrices. Furthermore, $\sigma\hat{M}_{12}H_{12}, \sigma\phi_{12}\hat{N}_{12}H_{12}$ are extended gradients (see Appendix A).

Since the above are gradients, $[\hat{M}^{(n)}H_{12}^{(1)}, \hat{N}^{(1)}H_{12}^{(2)}]_d = -\hat{M}^{(n+1)}H_{12}^{(3)}$ implies (3.11a). Then $[\hat{M}^{(n)}H_{12}^{(1)}, \hat{M}H_{12}^{(2)}] = -\hat{N}^{(n)}H_{12}^{(3)}$ implies (3.11b).

The above implies that the symmetries derived in Sect. II.C. give rise to conserved quantities. For example, the 1st and 2nd order t-dependent symmetries

$$\begin{aligned} \sigma_{12}^{(1)} &= \hat{M}_{12}^{(m)}H_{12}^{(1)} - 8\alpha t\hat{N}_{12}^{(m)}\cdot I, \\ \sigma_{12}^{(2)} &= \hat{N}_{12}^{(m)}H_{12}^{(2)} - t16\alpha\hat{M}_{12}^{(m)}H_{12}^{(1)} + t^2 64\alpha^2\hat{N}_{12}^{(m+2)}\cdot I, \end{aligned}$$

of the DS equation $Q_{1_t} = M_{11}^{(2)} = -[2\sigma(Q_{1_{xx}} + \alpha^2 Q_{1_{y_1y_1}}) - Q_1A_1 + A_1Q_1]$, $(D-JD_1)A_1 = -2(D+JD_1)\sigma Q_1^2$, obtained from Eqs. (2.29–30), correspond to the extended gradients $\sigma\sigma_{12}^{(1)}, \sigma\sigma_{12}^{(2)}$; then they give rise to the following t-dependent conserved quantities (see Eqs. (4.24)):

$$I^{(1)} = \int_{\mathbb{R}^2} dx\,dy \operatorname{trace}\sigma\left[Q_1, \frac{1}{2(m+1)}(D^{-1}\hat{M}_{12}^{(m+1)}H_{12}^{(1)})_{11} - \frac{t4\alpha}{m+1}\hat{N}_{12}^{(m+1)}\cdot I\right],$$

$$\begin{aligned} I^{(2)} = \int_{\mathbb{R}^2} dx\,dy_1 \operatorname{trace}\sigma\Big[Q_1, &\frac{1}{2(m+1)}(D^{-1}\hat{N}_{12}^{(m+1)}H_{12}^{(2)})_{11} \\ &- \frac{t8\alpha}{m+1}(D^{-1}\hat{M}_{12}^{(m+1)}H_{12}^{(1)})_{11} + \frac{t^2 32\alpha^2}{m+3}(D^{-1}\hat{N}_{12}^{(m+3)}\cdot I)_{11}\Big]. \end{aligned}$$

IV. On a Non-Gradient Master-Symmetry

In this section we make extensive use of the isomorphism between Lie and Poisson brackets. Hence it is useful to investigate the properties of

$$\Theta(g_L - g_L^*) = T_L + \Theta T_L^* \Theta^{-1}; \quad T \doteq \Theta g, \quad \Theta_L = 0. \tag{4.1}$$

Lemma 4.1. *Let*

$$S \doteq \Phi_L[T] + [\Phi, T_L], \tag{4.2}$$

with its adjoint

$$S^* = \Phi^*_L[T] + [T^*_L, \Phi^*]. \tag{4.3}$$

a) *If Φ is hereditary then*

$$\Phi^*_L[\Phi^n T] + [\Phi^n T]^*_L \Phi^* - \Phi^*(\Phi^n T)^*_L = S^* \Phi^{*n}. \tag{4.4}$$

b) *If Φ is factorizable in terms of compatible Hamiltonian operators,* i.e. *if $\Phi = \Omega\Theta^{-1}$, where $\Omega + v\Theta$ is a Hamiltonian operator, Θ is invertible and v is an arbitrary constant, then*

$$(\Phi T)_L + \Theta(\Phi T)^*_L \Theta^{-1} = \Phi(T_L + \Theta T^*_L \Theta^{-1}) + \Theta S^* \Theta^{-1}, \tag{4.5}$$

where we have assumed for simplicity that $\Theta_L = 0$.

c)

$$(\Phi^n T)_L + \Theta(\Phi^n T)^*_L \Theta^{-1} = \Phi^n(T_L + \Theta T^*_L \Theta^{-1}) + \sum_{r=1}^{n} \Phi^{r-1}\Theta\Phi^{*n-r} S^* \Theta^{-1}. \tag{4.6}$$

Proof. Equation (4.4) is the adjoint of (2.7) for $K = T$. Equation (4.5) is derived in Appendix B, and (4.6) follows from (4.5) by induction.

Theorem 4.1. *Assume that Φ is factorizable in terms of compatible Hamiltonian operators and that $\Theta_L = 0$. Further assume that $\Theta^{-1}\Phi^n M$ is a gradient function and that Φ is a strong symmetry for M. Then*

$$\begin{aligned}\Phi^m \sum_{r=1}^{n} \Phi^{n-r} S \Phi^{r-1} M &= \Theta \operatorname{grad}\langle \Theta^{-1}\Phi^n M, \Phi^m T\rangle \\ &\quad - \sum_{r=1}^{m} \Phi^{r-1}\Theta\Phi^{*m-r} S^* \Theta^{-1}\Phi^n M \\ &\quad - \Phi^m(T_L + \Theta T^*_L \Theta^{-1})\Phi^n M - \Phi^{n+m}[M, T]_L.\end{aligned} \tag{4.7}$$

Proof. Using the fact that $\Theta^{-1}\Phi^n M$ is a gradient, Eq. (3.2) becomes

$$[\Phi^n M, \Phi^m T]_L = \Theta \operatorname{grad}\langle \Theta^{-1}\Phi^n M, \Phi^m T\rangle - \{(\Phi^m T)_L + \Theta(\Phi^m T)^*_L \Theta^{-1}\}\Phi^n M. \tag{4.8}$$

Since M is a strong symmetry of Φ, Theorem 2.1 implies

$$[\Phi^n M, \Phi^m T]_L = \Phi^{n+m}[M, T]_L + \Phi^m\left(\sum_{r=1}^{n} \Phi^{n-r} S \Phi^{r-1}\right) M. \tag{4.9}$$

Using the above and (4.6) in (4.8) we obtain (4.7).

Equations (4.6) and (4.9) are useful in finding non-gradient master-symmetries for equations in 2 + 1. Furthermore, Theorem 4.1 is useful for deriving the potentials of various gradients. Formulae (4.6), (4.9) and (4.7) take a particularly simple form

if the function T_{12} is such that

i) $$S_{12} = S^*_{12} = c1, \tag{4.10a}$$

where 1 is the identity operator and c is an arbitrary constant, and

ii) $$T_{12_d} + \Theta_{12} T^*_{12_d} \Theta_{12}^{-1} = 0. \tag{4.10b}$$

In the following two examples the non-gradient master-symmetries are generated through functions T_{12} that satisfy Eqs. (4.10).

IVA. Equations Associated with the KP *Equation*

Corollary 4.1.

a) $\Phi^2_{12}\delta_{12}$ *is a non-gradient master-symmetry for the* KP *and the equations related to* KP:

$$[\Phi^n_{12}\hat{K}^0_{12}H_{12}, \Phi^2_{12}\delta_{12}]_d = b_n \Phi^{n+1}_{12}\hat{K}^0_{12}H_{12}, \tag{4.11}$$

$$8\Phi_{12} = (\Phi^2_{12}\delta_{12})_d + \Theta_{12}(\Phi^2_{12}\delta_{12})^*_d \Theta_{12}^{-1}, \tag{4.12}$$

where b_n *and* H_{12} *are given by*

$$b_n = 4n, \quad H_{12} = H(y_1, y_2) \text{ arbitrary, if } \hat{K}^0_{12} = \hat{N}_{12}, \tag{4.13a}$$

and by

$$b_n = 2(2n+1), \quad H_{12} = (y_1 + y_2)^r, \quad r = 0, 1, \quad \text{if } \hat{K}^0_{12} = \hat{M}_{12}. \tag{4.13b}$$

b) *Let*

$$\hat{\gamma}^{(n)}_{12} \doteq \Phi^{*n}_{12}\hat{\gamma}^0_{12}, \quad \hat{\gamma}^0_{12} \doteq \Theta_{12}^{-1}\hat{K}^0_{12}. \tag{4.14}$$

Then

$$\hat{\gamma}^{(n)}_{12} H_{12} = \mathrm{grad}_{12} I_n, \tag{4.15a}$$

$$I_n \doteq \frac{1}{b_{n+1}}\langle \hat{\gamma}^{(n+1)}_{12}H_{12}, \delta_{12}\rangle = \frac{1}{b_{n+1}}\int_{\mathbb{R}^3} dx\,dy_1\,dy_2\,\delta_{12}\hat{\gamma}^{(n+1)}_{12}H_{12}$$

$$= \frac{1}{b_{n+1}}\int_{\mathbb{R}^2} dx\,dy_1 (\hat{\gamma}^{(n+1)}_{12}H_{12})_{11}, \tag{4.15b}$$

where b_n *and* H_{12} *are given in (4.13).*

Proof. If

$$T_{12} = \delta_{12}, \tag{4.16}$$

Eq. (4.10b) is trivially satisfied and Eq. (4.10a) holds for $c = 4$, since $\Phi_{12_d}[\delta_{12}] = \Phi^*_{12_d}[\delta_{12}] = 4$. Equation (4.12) is a simple consequence of (4.6) for $n = 2$; using the following results

$$\Phi^n_{12}[\hat{N}_{12}H_{12}, \delta_{12}]_d = 0, \tag{4.17a}$$

$$\Phi^n_{12}[\hat{M}_{12}(y_1 + y_2)^r, \delta_{12}]_d = 2\Phi^{n-1}_{12}\hat{M}_{12}(y_1 + y_2)^r, \quad r = 0, 1, \tag{4.17b}$$

(see Appendix A) in Eqs. (4.9) and (4.7) (with $M = \hat{K}^0_{12}H_{12}$ and H_{12} as in (4.13)), we obtain

$$[\Phi^n_{12}\hat{K}^0_{12}H_{12}, \Phi^m_{12}\delta_{12}]_d = b_n \Phi^{n+m-1}_{12}\hat{K}^0_{12}H_{12} \tag{4.18}$$

(that reduces to (4.11) for $m = 2$), and

$$b_n \Phi_{12}^{*n+m-1} \hat{K}_{12}^0 H_{12} = \Theta_{12} \operatorname{grad}_{12} \langle \hat{\gamma}_{12}^{(n)} H_{12}, \Phi_{12}^m \delta_{12} \rangle, \tag{4.19}$$

where we have used $\Phi_{12}^n \Theta_{12} = \Theta_{12} \Phi_{12}^{*n}$. Equation (4.19) reduces to (4.15) if one uses the definition of $\langle f_{12}, g_{12} \rangle$ given by (1.20) and (3.4c).

Remark 4.1.

i) $T \doteq \Phi^2 1$ is a non-gradient master-symmetry for the KdV equation. Given T one recovers Φ from $T_f + \Theta T_f^* \Theta^{-1}$. Equation (4.12) is the two-dimensional analogue of this well known formula [8]–[10].

ii) Theorem 3.2 implies that Eqs. (4.15) with $m = 1$, $H_{12} = 1$ reduce to the following formula [6]:

$$\gamma_{11}^{(n)} = \frac{1}{b_{n+1}} \operatorname{grad} \int_{\mathbb{R}^2} dx\, dy_1 \gamma_{11}^{(n+1)}. \tag{4.20}$$

An analogous formula, for the KdV equation is well known

$$\gamma^{(n)} = \frac{1}{2(2n+3)} \operatorname{grad} \int_{\mathbb{R}} dx \gamma^{(n+1)}.$$

iii) We observe that Eq. (4.18) for $H_{12} = 1$ cannot be projected into Eq. (2.37).

IVB. Equations Associated with the DS *Equation*

Corollary 4.2.

a) $\Phi_{12}^2 T_{12}, T_{12} \doteq (x/2) \sigma Q_{12}^+ \delta_{12} I$, $I = \operatorname{diag}(1,1)$, *is a non-gradient master-symmetry for the* DS *and the equations related to* DS:

$$[\Phi_{12}^n \hat{K}_{12}^0 H_{12}, \Phi_{12}^2 T_{12}]_d = n \Phi_{12}^{n+1} \hat{K}_{12}^0 H_{12}, \tag{4.21}$$

$$2\Phi_{12} = (\Phi_{12}^2 T_{12})_d + \Theta_{12} (\Phi_{12}^2 T_{12})_d^* \Theta_{12}^{-1}, \quad \Theta_{12} = \sigma, \tag{4.22}$$

where $\hat{K}_{12}^0 H_{12}$ *is defined in (1.11–12).*

b) *Let*

$$\hat{\gamma}_{12}^{(n)} = \Phi_{12}^{*n} \hat{\gamma}_{12}^0, \quad \hat{\gamma}_{12}^0 = \Theta_{12}^{-1} \hat{K}_{12}^0, \quad \Theta_{12} = \sigma. \tag{4.23}$$

Then

$$\hat{\gamma}_{12}^{(n)} H_{12} = \operatorname{grad}_{12} I_n, \tag{4.24a}$$

$$I_n \doteq \frac{1}{n+1} \langle \hat{\gamma}_{12}^{(n+1)} H_{12}, T_{12} \rangle = -\frac{1}{2(n+1)} \int_{\mathbb{R}^3} dx\, dy_1\, dy_2 \operatorname{trace} \delta_{12} Q_{12}^+ \sigma \hat{\gamma}_{12}^{(n+1)} H_{12}$$

$$= \frac{1}{2(n+1)} \int_{\mathbb{R}^2} dx\, dy_1 \operatorname{trace} \sigma [Q_1, (\hat{\gamma}_{12}^{(n+1)} H_{12})_{11}]. \tag{4.24b}$$

Proof. If

$$T_{12} \doteq \frac{x}{2} \sigma Q_{12}^+ \delta_{12} I, \tag{4.25}$$

Eq. (4.10b) is satisfied and Eq. (4.10a) holds for $c = 1$ (see Appendix A). Then the

derivation of Eqs. (4.21), (4.22) and (4.24) is analogous to the one of Corollary 4.1. (see Appendix A).

V. 2 + 1 Dimensional Equations as Reductions of Non-Local Systems

In [1] and [14] the classes of evolution equations

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} \Phi_{12}^n \hat{K}_{12}^0 \cdot 1, \tag{5.1}$$

where Φ_{12} and $\hat{K}_{12}^0$ are defined in (1.4–5), were algorithmically derived from the spectral problem

$$w_{xx} + q(x,y)w + \alpha w_y = 0. \tag{5.2}$$

In this section we show that Eqs. (5.1) are exact reductions of equations non-local in y, generated by the following non-local analogue of (5.2):

$$w_{xx} + \tilde{q}w + \alpha w_y = \lambda w, \tag{5.3}$$

where

$$(\tilde{q}f)(x,y) \doteq \int_{\mathbb{R}} dy_2 q(x,y,y_2) f(x,y_2). \tag{5.4}$$

Hereafter the symbols $\tilde{u}$ and u_{12} indicate the integral operator defined by

$$(\tilde{u}f)(x,y) \doteq \int_{\mathbb{R}} dy_2 u(x,y,y_2) f(x,y_2) \tag{5.5}$$

and its kernel $u_{12} \doteq u(x,y_1,y_2)$, respectively.

The algorithmic derivation of the classes of evolution equations associated with (5.3) is standard; its main steps are:

i) Compatibility. A compatibility between Eq. (5.3), written in the more convenient form $\begin{pmatrix} w \\ w_x \end{pmatrix}_x = \begin{pmatrix} 0, & 1 \\ \lambda - \tilde{q} - D_y, & 0 \end{pmatrix} \begin{pmatrix} w \\ w_x \end{pmatrix}$, and the linear evolution equation $\begin{pmatrix} w \\ w_x \end{pmatrix}_t = \tilde{V} \begin{pmatrix} w \\ w_x \end{pmatrix}$, yields the following operator equation:

$$\begin{aligned} \tilde{q}_t = {} & \tilde{c}_{xxx} + [\tilde{q} + \alpha D_y, \tilde{c}]_x^+ + [\tilde{q} + \alpha D_y, \tilde{c}_x]^+ + [\tilde{q} + \alpha D_y, D^{-1}[\tilde{q} + \alpha D_y, \tilde{c}]] \\ & - 4\lambda \tilde{c}_x + \tilde{A}_0(\tilde{q} + \alpha D_y) - (\tilde{q} + \alpha D_y)\tilde{A}_0, \end{aligned} \tag{5.6}$$

where the scalar integral operator $2\tilde{c}$ is the 1,2 component of the 2×2 matrix integral operator $\tilde{V}, A_{0_x} = 0$ and [,] and $[\ ,\ ]^+$ are the usual commutator and anticommutator.

ii) Equation for the kernel. The operator equation (5.6), together with the definition (5.5), implies the following equation for the kernels q_{12}, c_{12}, A_{12}:

$$q_{12_t} = D\tilde{\Psi}_{12} C_{12} - \tilde{q}_{12}^- A_{12} + -4\lambda c_{12_x}, \tag{5.7}$$

where

$$\tilde{\Psi}_{12} \doteq D^2 + \tilde{q}_{12}^+ + D^{-1}\tilde{q}_{12}^+ D + D^{-1}\tilde{q}_{12}^- D^{-1}\tilde{q}_{12}^-, \tag{5.8a}$$

$$\tilde{q}_{12}^{\pm} f_{12} = \int_{\mathbb{R}} (q_{13} f_{32} \pm f_{13} q_{32}) dy_3 + \alpha(D_1 \mp D_2) f_{12}. \tag{5.8b}$$

iii) Expansion in powers of λ. Let us first assume that

$$C_{12} = \sum_{j=0}^{n} \lambda^j C_{12}^{(j)}, \quad A_{12} = 0, \tag{5.9}$$

equating the coefficients of $\lambda^j(0 \leqq j \leqq n)$ to zero we obtain: $C_{12}^{(n)} = H_{12}^{(n)}$; $C_{12}^{(j-1)} = \frac{1}{4}\tilde{\Psi}_{12}C_{12}^{(j)} + H_{12}^{(j-1)}(1 \leqq j \leqq n)$; $q_{12_t} = D\tilde{\Psi}_{12}C_{12}^{(0)}$; where $H_{12}^{(j)} = H^{(j)}(y_1, y_2)$. Then $C_{12}^{(0)} = \sum_{s=0}^{n} 4^{s-n}\tilde{\Psi}_{12}^{n-s}H_{12}^{(n-s)}$ and

$$q_{12_t} = \sum_{s=0}^{n} 4^{s-n} D\tilde{\Psi}_{12}^{n-s+1}H_{12}^{(n-s)} = \sum_{s=0}^{n} 4^{s-n}\tilde{\Phi}_{12}^{n-s+1}\cdot H_{12}^{(n-s)}, \tag{5.10}$$

where

$$\tilde{\Phi}_{12} \doteqdot D\tilde{\Psi}_{12}D^{-1} = D^2 + \tilde{q}_{12}^{+} + D\tilde{q}_{12}^{+}D^{-1} + \tilde{q}_{12}D^{-1}\tilde{q}_{12}D^{-1}. \tag{5.11}$$

If we assume that

$$C_{12} = \sum_{j=0}^{n} \lambda^j C_{12}^{(j)}, \quad A_{12} = -4\sum_{j=0}^{n+1} \lambda^j \bar{H}_{12}^{(j)}, \quad \bar{H}_{12}^{(j)} = \bar{H}^{(j)}(y_1, y_2),$$

then $C_{12}^{(n)} = D^{-1}\tilde{q}_{12}^{-}\cdot\bar{H}_{12}^{(n+1)} + H_{12}^{(n)}$; $C_{12}^{(j-1)} = \frac{1}{4}\tilde{\Psi}_{12}C_{12}^{(j)} + D^{-1}\tilde{q}_{12}\cdot H_{12}^{(j)} + H_{12}^{(j-1)}(1 \leqq j \leqq n)$; $q_{12_t} = D\tilde{\Psi}_{12}C_{12}^{(0)} + 4\tilde{q}_{12}^{-}\bar{H}_{12}^{(0)}$, where $H_{12}^{(j)} = H^{(j)}(y_1, y_2)$. The choice $H_{12}^{(j)} = 0$ for $0 \leqq j \leqq n$ yields $C_{12}^{(0)} = \sum_{s=0}^{n} 4^{s-n}\tilde{\Psi}_{12}^{n-s}D^{-1}\tilde{q}_{12}^{-}\cdot\bar{H}_{12}^{(n-s+1)}$ and

$$q_{12_t} = \sum_{s=0}^{n+1} 4^{s-n}D\tilde{\Psi}_{12}^{n-s+1}D^{-1}\tilde{q}_{12}^{-}\cdot\bar{H}_{12}^{(n-s+1)} = \sum_{s=0}^{n+1} 4^{s-n}\tilde{\Phi}_{12}^{n-s+1}\tilde{q}_{12}^{-}\cdot\bar{H}_{12}^{(n-s+1)}. \tag{5.12}$$

Thus the isospectral problem (5.3) generates the classes of evolution equations (5.10) and (5.12).

It turns out that the transformation $q_{12} \to \delta_{12}q_1$, $q_1 = q(x, y_1)$, is an exact reduction of Eqs. (5.10–11) if, at the same time, $4^{s-n}H_{12}^{(n-s)}, 4^{s-n}\bar{H}_{12}^{(n+1-s)} \to \beta^s\binom{n+1}{s}\delta_{12}^s$. In this case $\tilde{q}_{12}^{\pm} \to q_{12}^{\pm}$, $\tilde{\Phi}_{12} \to \Phi_{12}$ and

$$\delta_{12}q_{1_t} = \sum_{l=0}^{n+1}\beta^l\binom{n+1}{l}\Phi_{12}^{n+1-l}D\cdot\delta_{12}^l = \delta_{12}\Phi_{12}^{n+1}D\cdot 1 = \delta_{12}\Phi_{12}^n\hat{M}_{12}\cdot 1, \tag{5.13a}$$

$$\delta_{12}q_{1_t} = \sum_{l=0}^{n+1}\beta^l\binom{n+1}{l}\Phi_{12}^{n-l+1}q_{12}^{-}\cdot\delta_{12}^l = \delta_{12}\Phi_{12}^n q_{12}^{-}\cdot 1 = \delta_{12}\Phi_{12}^n\hat{N}_{12}\cdot 1. \tag{5.13b}$$

Proceeding exactly in the same way it is possible to show that the nonlocal eigenvalue problem

$$W_x = JW_y + \tilde{Q}W + \lambda JW, \tag{5.14}$$

generates the following class of evolution equations:

$$Q_{12_t} = \sum_{l=0}^{n} a_{n,l}\tilde{\Phi}_{12}^{n-l}\tilde{Q}_{12}^{-}\cdot H_{12}^{(l)}, \quad Q_{12} \doteqdot Q(x, y_1, y_2), \tag{5.15}$$

where

$$\tilde{\Phi}_{12}F_{12} \doteqdot \sigma(P_{12} - \tilde{Q}_{12}^{+}P_{12}^{-1}\tilde{Q}_{12}^{+})F_{12}, \quad F_{12} \doteqdot F(x, y_1, y_2) \text{ off-diagonal} \tag{5.16a}$$

$$\tilde{Q}_{12}^{\pm}F_{12} \doteqdot \int_{R} dy_3(Q_{13}F_{32} \pm F_{13}Q_{32}), \tag{5.16b}$$

$\sigma = \mathrm{diag}(1, -1)$ and $H_{12}^{(j)}$ is defined by

$$P_{12}H_{12}^{(j)} = 0, \quad H_{12}^{(j)} \text{ diagonal.} \tag{5.16c}$$

Also in this case the transformation $Q_{12} \to \delta_{12}Q_1$ is a reduction of (5.15) if $a_{n,l} \to \beta^l \binom{n}{l} (\beta = 2\alpha)$ and $H_{12}^{(l)} \to \delta_{12}^l I$ or $\delta_{12}^l \sigma$. In fact, $\tilde{Q}_{12}^{\,l} \to Q_{12}^{l}$, $\tilde{\Phi}_{12} \to \Phi_{12}$. Thus one obtains the following classes of equations:

$$\delta_{12}Q_{1_t} = \sum_{l=0}^{n} \beta^l \binom{n}{l} \Phi_{12}^{n-l} Q_{12}^{-} \delta_{12}^l I = \delta_{12}\Phi_{12}^n Q_{12}^{-} I \tag{5.17a}$$

or

$$\delta_{12}Q_{1_t} = \sum_{l=0}^{n} \beta^l \binom{n}{l} \Phi_{12} Q_{12}^{-} \delta_{12}^l \sigma = \delta_{12}\Phi_{12}^n Q_{12}^{-} \sigma, \tag{5.17b}$$

associated with the eigenvalue problem

$$W_x = JW_y + WQ + \lambda JW.$$

The above results clearly imply that all the notions introduced in [1] to characterize the algebraic properties of equations in $2+1$ dimensions can be justified and interpreted in terms of the algebraic structure of the corresponding non-local versions. For example:

i) The above derivations both motivate and explain the derivation of the recursion operators introduced in [1] and [14]. In particular the crucial role played by the integral representation of differential operators is clarified.

ii) The directional derivative introduced in [1], which is the main tool needed to investigate the algebraic properties of equations in $2+1$ dimensions, can be derived from the usual Frechét derivative in the space of non-local operators. For example, the Frechét derivative of $\tilde{q}_{12}^{\pm} g_{12}$ in a direction f_{12} is

$$\tilde{q}_{12}^{\pm}[f_{12}]g_{12} = f_{12}^{\pm} g_{12}, \tag{5.18a}$$

$$f_{12}^{\pm} g_{12} \doteq \int_{\mathbb{R}} dy_3 (f_{13}g_{32} \pm g_{13}f_{32}), \tag{5.18b}$$

which is exactly the direction derivative $q_{12_d}^{\pm}[f_{12}]g_{12}$ introduced in [1].

iii) The definition of an admissible function and of its derivative follows from the fact that reduced functions admit a double representation; for example (5.13b) implies

$$\sum_{l=0}^{n} \beta^l \binom{n}{l} \Phi_{12}^{n-l} q_{12}^{-} \delta_{12}^l = \delta_{12}\Phi_{12}^n q_{12}^{-} \cdot 1. \tag{5.19}$$

But the directional derivative is defined *only* on the admissible representation given by the left-hand side of (5.19), which is the form of the function before the reduction:

$$\sum_{l=0}^{n} a_{n,l} \tilde{\Phi}_{12}^{n-l} \tilde{q}_{12}^{-} H_{12}^{(l)}.$$

In Appendix A we investigate (Eqs. (A.3)) the algebra of the nonlocal operators $a_{12}^{\pm}$defined in (5.18b). Here we remark that this algebra can also be interpreted as an algebra of matrices in which $\pm$ indicates the operations of anticommutator

and commutater respectively, namely $a^{\pm} b = ab \pm ba$. (See also Appendix C of [1].) This is not a coincidence and the following important observations, here illustrated on the recursion operator Φ_{12} of the KP class, can be made.

i) Integral operators:

$$q_{12}^{\pm} f_{12} = \int_{\mathbf{R}} dy_3 (q_{13} f_{32} \pm f_{13} q_{32}), \tag{5.20a}$$

$$q_{12} = \delta_{12} q_1 + \alpha \delta'_{12}, \tag{5.20b}$$

is equivalent to the introduction of the integral operator $\tilde{q}_{12}^{\pm}$. Then Φ_{12} becomes the nonlocal recursion operator $\tilde{\Phi}_{12}$, defined in (5.11) and associated with the nonlocal eigenvalue problem (5.3).

ii) Matrix operators:

$$q^{\pm} f \doteq qf \pm fq; \quad q, f \text{ matrices}, \tag{5.21}$$

reduces Φ_{12} to the well-known matrix recursion operator

$$\Phi \doteq D^2 + q^{+} + Dq^{+} D^{-1} + q^{-} D^{-1} q^{-} D^{-1}, \tag{5.22}$$

associated with the $N \times N$ matrix Schroedinger eigenvalue problem in one dimension [15].

The directional derivative $q_{12_d}^{\pm}[f_{12}] g_{12}$ of $q_{12}^{\pm}$:

$$q_{12_d}^{\pm}[f_{12}] g_{12} = f_{12}^{\pm} g_{12}, \tag{5.23}$$

i) is exactly the usual Fréchet derivative $\tilde{q}_{12}^{\pm\prime}[f_{12}] g_{12}$ of $\tilde{q}_{12}^{\pm}$.

ii) Corresponds to the usual Fréchet derivative $q^{\pm\prime}[f] g$ of $q^{\pm}$:

$$q^{\pm\prime}[f] g = f^{\pm} g = fg \pm gf. \tag{5.24}$$

Since the $\pm$ operators in (5.20a), (5.8b), (5.21) and (5.18b) satisfy the same algebraic identities (A.3), then important algebraic properties of the recursion operator Φ_{12} of the KP equation (like hereditariness) are equivalent to the corresponding properties of the nonlocal recursion operator $\tilde{\Phi}_{12}$ (5.11) and, even more remarkable, of the matrix recursion operator Φ_{12} (5.22).

In order to make this connection with the matrix formalism more clear, we observe that the nonlocal problem (5.3) can be obtained taking the $N \to \infty$ limit of the $N \times N$ matrix one dimensional Schroedinger problem

$$\underline{W}_{xx} + \underline{q}\,\underline{W} = \lambda \underline{W}, \tag{5.25}$$

where the coefficients of the matrix $\underline{q}$ are chosen in the form

$$(\underline{q})_{ij} = q_{ij}(x,t) + a(\delta_{i,j+1} - \delta_{i,j-1}), \tag{5.26}$$

with the obvious prescriptions

$$q_{ij}(x,t) \xrightarrow[N \to \infty]{} q(x,t,y_1,y_2); \quad a(\delta_{i,j+1} - \delta_{i,j-1}) \xrightarrow[N \to \infty]{} \alpha \frac{\partial}{\partial y_1}. \tag{5.27}$$

The connection between equations in $2+1$ and $N \times N$ matrix equations in $1+1$ was first used by P. Caudrey. He introduced in [16] a $N \times N$ spectral problem

(similar to (5.25)) which reduces to (5.2) in the limit $N \to \infty$. Then he showed that the $N \times N$ Riemann–Hilbert formalism associated with it becomes, in the limit $N \to \infty$, the nonlocal Riemann–Hilbert and the $\bar{\partial}$ formalisms of (5.2) [17].

The connection established in this section between the spectral problems (5.25), (5.3) and (5.2) implies that the well established theory of recursion operators and their connection to the bi-Hamiltonian formalism in 1 + 1 dimensions, once applied to the matrix problem (5.25), gives rise, in the limit $N \to \infty$, to the corresponding theory developed in [1] and this paper for 2 + 1 dimensional systems.

It is remarkable that both algebraic properties and methods of solution for integrable systems in 2 + 1 dimensions can be justified and obtained from the corresponding properties of 1 + 1 dimensional systems.

Appendix A

In this Appendix we present some of the explicit calculations necessary to apply the results presented in this paper to the classes of evolution equations associated with the KP and the DS equations. In order to make this paper self-contained, we first present some results contained in Appendices B, C of [1].

The directional derivatives of the basic operators $q_{12}^{\pm}$ and $Q_{12}^{\pm}$, defined in (1.4b) and (1.10b) respectively, are

$$q_{12_d}^{\pm}[f_{12}]g_{12} = f_{12}^{\pm}g_{12}, \quad f_{12}, g_{12} \text{ scalars}, \tag{A.1a}$$

$$Q_{12_d}^{\pm}[f_{12}]g_{12} = f_{12}^{\pm}g_{12}, \quad f_{12} \text{ off-diagonal matrix}, \tag{A.1b}$$

where $f_{12}^{\pm}$ are defined by

$$f_{12}^{\pm}g_{12} \doteq \int_{\mathbb{R}} dy_3(f_{13}g_{32} \pm g_{13}f_{32}). \tag{A.2}$$

The integral operators $f_{12}^{\pm}$ have the following algebraic properties:

$$a_{12}^{\pm}b_{12} = \pm b_{12}^{\pm}a_{12}, \tag{A.3a}$$

$$(a_{12}^{+}b_{12}^{+} - b_{12}^{+}a_{12}^{+})c_{12} = (a_{12}^{-}b_{12})^{-}c_{12} = -c_{12}^{-}a_{12}^{-}b_{12}, \tag{A.3b}$$

$$(a_{12}^{+}b_{12}^{-} \mp b_{12}^{\pm}a_{12}^{\pm})c_{12} = (a_{12}^{\pm}b_{12})^{\pm}c_{12} = \pm c_{12}^{\pm}a_{12}^{\pm}b_{12}, \tag{A.3c}$$

$$a_{12}^{\pm *} = \pm a_{12}^{\pm}. \tag{A.3d}$$

Moreover the integral representations

$$q_{12}^{\pm}f_{12} = \int_{\mathbb{R}} dy_3(q_{13}f_{32} \pm f_{13}q_{32}), \quad q_{12} = \delta_{12}q_1 + \alpha\delta'_{12},$$

$$Q_{12}^{\pm}f_{12} = \int_{\mathbb{R}} dy_3(Q_{13}f_{32} \pm f_{13}Q_{32}), \quad Q_{12} = \delta_{12}Q_1,$$

imply that the operators $q_{12}^{\pm}$ and $Q_{12}^{\pm}$ satisfy Eqs. (A.3) as well. Equations (A.3) are conveniently used to show that:

a) The recursion operators Φ_{12} (1.4) and (1.10) are strong symmetries of the starting symmetries $\hat{K}_{12}^{0}H_{12}$ (1.5–6) and (1.11–12) respectively. For example, if

$\hat{K}^0_{12} = Q^-_{12}$ and H_{12} is given by (1.12),

$$\begin{aligned}\Phi_{12_d}[Q^-_{12}H_{12}]f_{12} &- (Q^-_{12}H_{12})_d[\Phi_{12}f_{12}] + \phi_{12}(Q_{12}H_{12})_d[f_{12}]\\ &= -\sigma[(Q^-_{12}H_{12})^+ P^{-1}_{12}Q^+_{12} + Q^+_{12}P^{-1}_{12}(Q^-_{12}H_{12})^+]f_{12}\\ &\quad -(\sigma(P_{12} - Q^+_{12}P^{-1}_{12}Q^+_{12})f_{12})^- H_{12} + \sigma(P_{12} - Q^+_{12}P^{-1}_{12}Q^+_{12})f^-_{12}H_{12} = 0,\end{aligned}$$

since the terms without $Q^{\dagger}_{12}$ give

$$-\sigma(P_{12}f_{12})^- H_{12} + \sigma P_{12}f^-_{12}H_{12} = 0,$$

and the terms with Q^+_{12} give

$$\begin{aligned}-\sigma(Q^-_{12}H_{12})^+ P^{-1}_{12}Q^+_{12}f_{12} &- \sigma Q^+_{12}P^{-1}_{12}(Q^-_{12}H_{12})^+ f_{12}\\ &+ (\sigma Q^+_{12}P^{-1}_{12}Q^+_{12}f_{12})^- H_{12} - \sigma Q^+_{12}P^{-1}_{12}Q^+_{12}f^-_{12}H_{12} = -\sigma(((Q^-_{12}H_{12})^+\\ &+ H^-_{12}Q^+_{12})P^{-1}_{12}Q^+_{12}f_{12} + Q^+_{12}P^{-1}_{12}(f^+_{12}Q^-_{12}H_{12} + Q^+_{12}f^-_{12}H_{12}))\\ &= -\sigma Q^+_{12}P^{-1}_{12}(H^-_{12}Q^+_{12}f_{12} + f^+_{12}Q^-_{12}H_{12} + Q^+_{12}f^-_{12}H_{12}) = 0.\end{aligned}$$

b) The Lie algebra of the starting symmetries is given by Eqs. (1.7) and (1.13). For example

i) if $\hat{K}^0_{12}H_{12}$ are given by (1.5–6):

$$\begin{aligned}[\hat{N}_{12}H^{(1)}_{12}, \hat{M}_{12}H^{(2)}_{12}]_d &= ((Dq^+_{12} + q^-_{12}D^{-1}q^-_{12})H^{(2)}_{12})^- H^{(1)}_{12} - D(q_{12}H^{(1)}_{12})^+ H^{(2)}_{12}\\ &\quad -(q^-_{12}H^{(1)}_{12})^- D^{-1}q^-_{12}H^{(2)}_{12} - q_{12}D^{-1}(q^-_{12}H^{(1)}_{12})^- H^{(2)}_{12}\\ &= -Dq^+_{12}(H^{(1)}_{12})^- H^{(2)}_{12} + q^-_{12}D^{-1}(-(H^{(1)}_{12})^- q^-_{12}H^{(2)}_{12}\\ &\quad +(H^{(2)}_{12})^- q_{12}H^{(1)}_{12}) = -\hat{M}_{12}H^{(3)}_{12};\end{aligned}$$

ii) if $\hat{K}^0_{12}H_{12}$ are given by (1.11–12)

$$\begin{aligned}[\hat{N}_{12}H^{(1)}_{12}, \hat{M}_{12}H^{(2)}_{12}]_d &= (Q^-_{12}\sigma H^{(2)}_{12})^- H^{(1)}_{12} - (Q^-_{12}H^{(1)}_{12})^- \sigma H^{(2)}_{12}\\ &= -(H^{(1)}_{12})^- Q_{12}\sigma H^{(2)}_{12} + (\sigma(H^{(2)}_{12}))^- Q_{12}H^{(1)}_{12}\\ &= -\hat{M}_{12}H^{(3)}_{12}.\end{aligned}$$

c) The functions T_{12} given by (4.16) and (4.25) satisfy Eqs. (4.10); for examples

i) if $T_{12} = \delta_{12}$, then

$$S_{12}f_{12} = \Phi_{12_d}[\delta_{12}]f_{12} = (2\delta^+_{12} + \delta_{12}D^{-1}q^-_{12} + q^-_{12}D^{-1}\delta_{12})f_{12} = 4f_{12},$$

since $\delta_{12_d} = 0$ and $\delta^+_{12}f_{12} = 2f_{12}, \delta^-_{12}f_{12} = 0$.

ii) If $T_{12} = (x/2)\sigma Q^+_{12}\delta_{12}I$, then Eqs. (4.10) are satisfied using the following results:

$$T_{12_d}[f_{12}] = \frac{x}{2}\sigma f^+_{12}\delta_{12}I = x\sigma f_{12},$$

$$T^+_{12}f_{12} = x(\sigma Q_1 f_{12} \pm f_{12}\sigma Q_2) = x\sigma\begin{cases} Q^-_{12}f_{12}, & f_{12} \text{ off-diagonal}\\ Q^+_{12}f_{12}, & f_{12} \text{ diagonal}\end{cases}.$$

For instance:

$$\begin{aligned}S_{12}f_{12} = &-\sigma(T^+_{12}P^{-1}_{12}Q^+_{12} + Q^+_{12}P^{-1}_{12}T^+_{12})f_{12} + \sigma(P_{12} - Q^+_{12}P^{-1}_{12}Q^+_{12})x\sigma f_{12}\\ &- x(P_{12} - Q^+_{12}P^{-1}_{12}Q^+_{12})f_{12} = f_{12}.\end{aligned}$$

d) $\Phi_{12}^n[\hat{K}_{12}^0 H_{12}, T_{12}] = 0$, if $\hat{K}_{12}^0 H_{12}$ and T_{12} are given by (1.11–12) and (4.25) respectively, or by $q_{12}^- H_{12}, H_{12} = H(y_1, y_2)$, and δ_{12}. For example

i) $\Phi_{12}^n[q_{12}^- H_{12}, \delta_{12}]_d = \Phi_{12}^n \delta_{12}^- \cdot H_{12} = 0.$

ii) $$\Phi_{12}^n[Q_{12}^- H_{12}, T_{12}]_d = \Phi_{12}^n(T_{12}^- H_{12} - T_{12_d}[Q_{12}^- H_{12}]) = \Phi_{12}^n(x\sigma Q_{12}^- - x\sigma Q_{12}^-)\cdot H_{12} = 0.$$

e) Equation (4.17b) holds. It follows from $\hat{M}_{12_d}[\delta_{12}] = D\delta_{12}^+ + \delta_{12}^- D^{-1} q_{12}^- + q_{12}^- D^{-1}\delta_{12}^- = 2D$, which implies

$$\Phi_{12}^n[\hat{M}_{12}\cdot H_{12}, \delta_{12}]_d = 2\Phi_{12}^n D\cdot H_{12}. \tag{A.4}$$

Different choices of $H_{12} = H(y_1, y_2)$ give different results. As it was shown in Appendix B of [1]

$$\Phi_{12}^{n+1} D\cdot H_{12} = \sum_{l=0}^{n} (2\alpha)^l \Phi_{12}^{n-l} \hat{M}_{12}\cdot H_{12}^{(l)}, \quad H_{12}^{(l)} \doteq \partial^l \frac{H(y_1 - y_2)}{\partial y_1^l}; \tag{A.5}$$

an analogous, although more tedious derivation, gives

$$\Phi_{12}^{n+1} D\cdot H_{12} = \Phi_{12}^n \hat{M}_{12}\cdot H_{12} + \sum_{l=1}^{v_n} a_l(2\gamma)^{2l} \Phi_{12}^{n-2l} \hat{M}_{12}\cdot H_{12}^{(2l)}, \tag{A.6a}$$

$$H_{12}^{(l)} \doteq \frac{\partial^l H(y_1 + y_2)}{\partial y_1^l}, \quad a_l \doteq C_l^{(l)}, \quad v_n \doteq \begin{cases} (n-1)/2, & n \text{ odd} \\ n/2, & n \text{ even} \end{cases} \tag{A.6b}$$

and the coefficients $C_l^{(l)}$ are obtained through the following recursive construction:

$$\begin{aligned} C_l^{(m)} &= C_l^{(m-1)} + 2C_{l-1}^{(m-1)} + C_{l-2}^{(m-1)}, \\ C_0^{(0)} &= 1, \end{aligned} \tag{A.7}$$

where $C_b^{(a)} = 0$ if $b < 0$ and $b > a$. Equations (A.4) and (A.6) imply Eq. (4.17b).

f) $\Theta_{12}^{-1}\Phi_{12}^n \hat{K}_{12}^0 H_{12}$ are extended gradients; for example if

i) $\hat{K}_{12}^0 = \hat{N}_{12} \doteq q_{12}^-, H_{12} = H(y_1, y_2), \Theta_{12} = D$ and $n = 0$:

$$\begin{aligned} \langle f_{12}, (D^{-1}\hat{N}_{12}H_{12})_d[g_{12}]\rangle &= \langle f_{12}, D^{-1} g_{12}^- H_{12}\rangle = \langle D^{-1} f_{12}, H_{12}^- g_{12}\rangle \\ &= -\langle H_{12}^- D^{-1} f_{12}, g_{12}\rangle = \langle D^{-1} f_{12}^- H_{12}, g_{12}\rangle. \end{aligned}$$

ii) $\hat{K}_{12}^0 = \hat{M}_{12} \doteq Dq_{12}^+ + q_{12}^- D^{-1} q_{12}^-, H_{12} = H(y_1, y_2), \Theta_{12} = D$ and $n = 0$:

$$\begin{aligned} &\langle f_{12}, (D^{-1}\hat{M}_{12}H_{12})_d[g_{12}]\rangle \\ &= \langle f_{12}, g_{12}^+ H_{12} + D^{-1} g_{12}^- D^{-1} q_{12}^- H_{12} + D^{-1} q_{12}^- D^{-1} g_{12}^- H_{12}\rangle \\ &= \langle f_{12}, (H_{12}^+ - D^{-1}((D^{-1} q_{12}^- H_{12})^- + q_{12}^- D^{-1} H_{12}^-)) g_{12}\rangle \\ &= \langle (H_{12}^+ - ((D^{-1} q_{12}^- H_{12})^- + H_{12}^- D^{-1} q_{12}^-) D^{-1}) f_{12}, g_{12}\rangle \\ &= \langle (H_{12}^+ - D^{-1}((D^{-1} q_{12}^- H_{12})^- + q_{12}^- H_{12}^- D^{-1})) f_{12}, g_{12}\rangle. \end{aligned}$$

iii) $\hat{K}_{12}^0 = \hat{M}_{12} \doteq Q_{12}^- \sigma, H_{12}$ defined in (1.12) and $n = 0$:

$$\langle f_{12}, (\sigma\hat{M}_{12}H_{12})_d g_{12}\rangle = \langle f_{12}, -H_{12}^+ g_{12}\rangle = \langle -H_{12}^+ f_{12}, g_{12}\rangle.$$

iv) $\hat{K}_{12}^0 = \hat{N}_{12} \doteq Q_{12}^-, H_{12}$ defined in (1.12) and $n = 1$:

$$\begin{aligned}&\langle f_{12},(\sigma\hat{N}^{(1)}_{12}H_{12})_a g_{12}\rangle\\&=\langle f_{12},(-(P^{-1}_{12}Q^{+}_{12}Q^{-}_{12}H_{12})^{+}-P_{12}H^{-}_{12}+Q^{+}_{12}H^{-}_{12}P^{-1}_{12}Q^{+}_{12})g_{12}\rangle\\&=\langle(-(P^{-1}_{12}Q^{+}_{12}Q^{-}_{12}H_{12})^{+}-P_{12}H^{-}_{12}+Q^{+}_{12}H^{-}_{12}P^{-1}_{12}Q^{+}_{12})f_{12},g_{12}\rangle.\end{aligned}$$

g) Equation (4.24b) holds, since

$$\begin{aligned}\langle\hat{\gamma}^{(n+1)}_{12}H_{12},x\sigma Q^{+}_{12}\delta_{12}I\rangle&=-\langle xQ^{+}_{12}\sigma\hat{\gamma}^{(n+1)}_{12}H_{12},\delta_{12}I\rangle\\&=\int_{\mathbb{R}^3}dx\,dy_1\,dy_2\,\delta_{12}\,\mathrm{trace}\,Q^{+}_{12}\sigma\hat{\gamma}^{(n+1)}_{12}H_{12}.\end{aligned}$$

Appendix B

In this Appendix we show that if Φ is factorizable in terms of compatible Hamiltonian operators Ω and Θ in the form $\Phi=\Omega\Theta^{-1}$, and if Θ is invertible and $\Theta_L=0$, then Eq. (4.5) holds.

We first show that

$$(\Phi T)^{*}_{L}=\mathscr{L}^{*}_{T}+T^{*}_{L}\Phi^{*},\quad \mathscr{L}_T b\doteq\Phi_L[b]T, \tag{B.1}$$

$$\Phi_L[v]T+\Theta\mathscr{L}^{*}_{T}\Theta^{-1}v=\Phi_L[T]b. \tag{B.2}$$

(B.1) simply follows from the definition of the adjoint:

$$\langle(\Phi T)^{*}_{L}a,b\rangle=\langle a,\Phi_L[b]T+\Phi T_L[b]\rangle=\langle(\mathscr{L}^{*}_{T}+T^{*}_{L}\Phi^{*})a,b\rangle,$$

while (B.2) requires the use of all the hypothesis of this Lemma.:

$$\begin{aligned}&\langle\Phi_L[v]T+\Theta\mathscr{L}^{*}_{T}\Theta^{-1}v,\alpha\rangle\\&=\langle\Omega_L[\Theta(\Theta^{-1}v)]\Theta^{-1}T,\alpha\rangle+\langle\Omega_L[\Theta\alpha]\Theta^{-1}v,\Theta^{-1}T\rangle\\&=\langle\alpha,\Omega_L[T]\Theta^{-1}v\rangle=\langle\alpha,\Phi_L[T]v\rangle.\end{aligned}$$

Then, using (B.1–2) and (4.4) for $n=0$, we obtain Eq. (4.5):

$$\begin{aligned}((\Phi T)_L+\Theta(\Phi T)^{*}_{L}\Theta^{-1})v&=\Phi(T_L[v]+\Theta T^{*}_{L}\Theta^{-1}v)+\Phi_L[v]T\\&\quad+\Theta\mathscr{L}^{*}_{T}\Theta^{-1}v+\Theta(T^{*}_{L}\Phi^{*}-\Phi^{*}T^{*}_{L})\Theta^{-1}v\\&=\Phi(T_L[v]+\Theta T^{*}_{L}\Theta^{-1}v)+\Theta S^{*}\Theta^{-1}v.\end{aligned}$$

Acknowledgements. It is a pleasure to acknowledge useful discussions with M. J. Ablowitz. One of the authors (P.M.S.) wishes to acknolwedge the warm hospitality of the Mathematics Department of the University of Paderborn, where the last part of this paper was completed. He also acknowledges interesting discussions with B. Fuchssteiner and W. Oevel. Particular thanks go to W. Oevel for generously computing some of the functions $\Phi^{m}_{12}\hat{K}^{0}_{12}H^{(r)}_{12}$ discussed in this paper, using the system of algebraic manipulation MACSYMA. This work was supported by the National Science Foundation under grant number DMS 8501325 and the Office of Naval Research under grant number N00014-76-C-0867.

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Communicated by A. Jaffe

Received December 30, 1986; in revised form August 10, 1987

Commun. Math. Phys. 115, 375–419 (1988)

Communications in
Mathematical
Physics

Recursion Operators and Bi-Hamiltonian Structures in Multidimensions. I

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Abstract. The algebraic properties of exactly solvable evolution equations in one spatial and one temporal dimensions have been well studied. In particular, the factorization of certain operators, called recursion operators, establishes the bi-Hamiltonian nature of all these equations. Recently, we have presented the recursion operator and the bi-Hamiltonian formulation of the Kadomtsev-Petviashvili equation, a two spatial dimensional analogue of the Korteweg-deVries equation. Here we present the general theory associated with recursion operators for bi-Hamiltonian equations in two spatial and one temporal dimensions. As an application we show that general classes of equations, which include the Kadomtsev-Petviashvili and the Davey-Stewartson equations, possess infinitely many commuting symmetries and infinitely many constants of motion in involution under two distinct Poisson brackets. Furthermore, we show that the relevant recursion operators naturally follow from the underlying isospectral eigenvalue problems.

1. Introduction

In recent years a deep connection has been discovered [1, 2] between certain nonlinear evolution equations in $1+1$, i.e. in one spatial and one temporal dimensions, and certain linear isospectral eigenvalue (or scattering) equations. These isospectral problems play a central role in developing methods for solving several types of initial value problems of the associated nonlinear evolution equations. The most well known such method, the celebrated inverse scattering transform (IST) method, deals with initial data decaying at infinity. However, the isospectral problem is also crucial for characterizing periodic [3] as well as self similar solutions [4].

It is quite satisfying, from a unified point of view, that the isospectral problems are also central in investigating the "algebraic" properties of the associated

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nonlinear evolution equations: The isospectral problem algorithmically implies a certain linear integrodifferential operator Φ, called the recursion operator. This operator has remarkable properties: Φ maps symmetries into symmetries; Φ has a certain algebraic property [5] which Fuchssteiner [6] calls hereditary and thus generates commuting symmetries; Φ^*, the adjoint of Φ, maps gradients of conserved quantities into gradients of conserved quantities; Φ, admits a symplectic-cosymplectic factorization and thus generates constants of motion in involution [7]; Φ times the first Hamiltonian operator produces the second Hamiltonian [8], hence the associated nonlinear evolution equations are bi-Hamiltonian systems; the eigenfunctions of Φ are also symmetries, which actually characterize the N-soliton solutions [9]; the eigenfunctions of Φ form a complete set [10].

Well-known scattering problems in $1+1$ are the Schrödinger scattering problem, the so-called generalized Zakharov-Shabat (ZS) or Ablowitz-Kaup-Newell-Segur (AKNS) system, and their natural generalization, i.e. the Gel'fand-Dikii operator, and the $N \times N$ AKNS. These isospectral problems are related to several physically important equations, the Korteweg-deVries (KdV), sine-Gordon, nonlinear Schrödinger, modified KdV, Boussinesq, N-wave interaction equations, etc. The above eigenvalue problems have been thoroughly investigated with respect to both the IST method and the associated algebraic properties. The IST of the Schrödinger was investigated in [1, 11], of the AKNS in [12], of the $N \times N$ AKNS in [13–15], and of the Gel'fand-Dikii in [16]. The IST of special important cases of the above systems were investigated in [17] (nonlinear Schrödinger), [18] (modified KdV), [19, 20] (Boussinesq), [21] (3-wave interactions). The recursion operator associated with the Schrödinger equation was obtained by Lenard, of the AKNS in [12], of the Gel'fand-Dikii in [22] and of the $N \times N$ AKNS in [5] and [23]. The general theory of recursion operators and their connection to bi-Hamiltonian formulation has been developed by Magri [8], Gel'fand and Dorfman [24], and Fokas and Fuchssteiner [7]. Other relevant works include [25].

It is also well known that certain two-dimensional generalizations of the above scattering equations are related to physically interesting nonlinear evolution equations in $2+1$ dimensions. In particular, a generalization of the Schrödinger equation is related to the Kadomtsev-Petviashvili (KP) equation (a two-dimensional analogue of the KdV). Similarly, the two-dimensional version of the $N \times N$ AKNS is related to N-wave interactions in $2+1$, the Davey-Stewartson equation (DS) (a two-dimensional analogue of the nonlinear Schrödinger) and the modified KP equation. The IST for the above two scattering problems has been only recently studied [26]. (For other interesting results in this direction see also [27].) In spite of this success, the question of using the scattering equations to obtain recursion operators had remained open. Actually, Zakharov and Konopelchenko [28] have shown that recursion operators of a certain type, naturally motivated from the results in $1+1$, do not in general exist in multidimensions. Recursion operators in $2+1$ dimensions were only known for straightforward examples like the $2+1$ dimension Burgers equation, that can be linearized via a generalized Cole-Hopf transformation [30b]. For a brief review of the literature of the various attempts to obtain recursion operators in $2+1$, we refer the reader to [29]. Here we only note that Konopelchenko and Dubrovsky [30a] were the first

to establish the importance of working with $w(x, y_1)w^+(x, y_2)$, as opposed to $w(x, y)w^+(x, y)$, where $w(x, y)$ and $w^+(x, y)$ denote the eigenfunctions of the associated scattering problem and of its adjoint, respectively. They also found a linear equation satisfied by $w(x, y_1)w^+(x, y_2)$. However, they failed to recognize that this equation could actually yield the recursion operator of the entire associated hierarchy of nonlinear equations. Instead, they used the above equation to obtain "local" recursion operators. Thus, the question of studying the remarkably rich structure of the recursion operator, in particular, its connection to symmetries, conservation laws and bi-Hamiltonian operators was not even posed.

Using a suitable generalization, we have recently presented the recursion operator and the two Hamiltonian operators associated with the KP equation [29]. In this paper we present the theory associated with these operators. In particular, the notions of symmetries, gradients of conserved quantities, strong and hereditary symmetries, Hamiltonian operators are generalized to equations in 2+1. Also a simple algorithmic approach is given for obtaining the recursion operator from the scattering problem. As examples of the above theory we study the two-dimensional Schrödinger problem and the 2×2 AKNS problem in two spatial dimensions. The following concrete results are given:

i) The linear eigenvalue problem

$$w_{xx} + q(x, y)w + \alpha w_y = 0, \tag{1.1}$$

where α is a constant parameter, gives rise to the *hereditary* recursion operator

$$\Phi_{12} = D^2 + q_{12}^+ + Dq_{12}^+ D^{-1} + q_{12}^- D^{-1} q_{12}^- D^{-1}, \tag{1.2a}$$

where the operators $q_{12}^\pm$ are defined by

$$q_{12}^\pm \doteq q_1 \pm q_2 + \alpha(D_1 \mp D_2), \quad D_i \doteq \frac{d}{dy_i}, \quad q_i \doteq q(x, y_i), \quad i = 1, 2. \tag{1.2b}$$

The operator Φ_{12} admits a factorization in terms of compatible Hamiltonian operators, $\Phi_{12} = \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$, where $\Theta_{12}^{(1)} = D$ and $\Theta_{12}^{(2)}$ are skew symmetric operators satisfying an appropriate Jacobi identity.

The KP equation

$$q_t = q_{xxx} + 6qq_x + 3\alpha^2 D^{-1} q_{yy} \tag{1.3}$$

is the second member, $n = 1$ $(\beta_1 = 1/2)$ of the following hierarchy generated by Φ_{12}

$$q_{1_t} = \beta_n \int_{-\infty}^{\infty} dy_2 \delta(y_1 - y_2) \Phi_{12}^n \sigma_{12}^{(0)}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where $\sigma_{12}^{(0)} = (\Phi_{12} D) \cdot 1 = q_{1_x} + q_{2_x} + (q_1 - q_2)D^{-1}(q_1 - q_2) + \alpha D^{-1}(q_{1_{y_1}} - q_{2_{y_2}})$ and $\delta(y_1 - y_2)$ is the Dirac delta function. The KP is a bi-Hamiltonian system:

$$q_{1_t} = \int_{-\infty}^{\infty} dy_2 \delta(y_1 - y_2) \Theta_{12}^{(1)} \gamma_{12}^{(1)} = \int_{-\infty}^{\infty} dy_2 \delta(y_1 - y_2) \Theta_{12}^{(2)} \gamma_{12}^{(0)}, \tag{1.5}$$

where

$$\gamma_{12}^{(0)} = D^{-1} \sigma_{12}^{(0)}, \quad \gamma_{12}^{(1)} = D^{-1} \Phi_{12} \sigma_{12}^{(0)}. \tag{1.6}$$

The KP equation possesses two infinite hierarchies of time-independent commuting symmetries and constants of motion. For example, $(\Phi^n_{12}\sigma^{(0)}_{12})_{11}$, $n=0,1,2,\ldots$ are symmetries of the KP.

The operator Φ_{12} is the adjoint with respect to an appropriate bilinear form (see Sect. 4) of the "squared eigenfunction" operator. One may verify that

$$\Phi^*_{12}w_1w_2^+=0, \quad w_i \doteq w(x,y_i) \tag{1.7}$$

where w^+ satisfies the adjoint of Eq. (1.1) (see Sect. 4).

ii) The linear eigenvalue problem

$$W_x=JW_y+QW, \tag{1.8}$$

where $J=\alpha\sigma$, $\sigma=\mathrm{diag}(1,-1)$, and Q is a 2×2 off-diagonal matrix containing the potentials $q_1(x,y)$, $q_2(x,y)$, gives rise to the hereditary recursion operator Φ_{12} defined on off-diagonal matrices, where

$$\Phi_{12} \doteq \sigma(P_{12}-Q^+_{12}P^{-1}_{12}Q^+_{12}), \tag{1.9a}$$

and the operators P_{12}, $Q^\pm_{12}$ are defined by

$$P_{12}F_{12} \doteq F_{12_x}-JF_{12_{y_1}}-F_{12_{y_2}}J, \quad Q^\pm_{12}F_{12} \doteq Q_1F_{12} \pm F_{12}Q_2, \tag{1.9b}$$

and $Q_i \doteq Q(x,y_i)$, $i=1,2$. The operator Φ_{12} admits a factorization in terms of Hamiltonian operators, $\Phi_{12}=\Theta^{(2)}_{12}(\Theta^{(1)}_{12})^{-1}$, where $\Theta^{(1)}_{12}=\sigma$.

The DS equation

$$iq_t+\tfrac{1}{2}(q_{xx}+\alpha^2q_{yy})=q(\phi-|q|^2); \quad \phi_{xx}-\alpha^2\phi_{yy}=2|q|^2_{xx}, \tag{1.10}$$

corresponds to $q_2=\bar{q}_1=\bar{q}$, $\beta_2=-\dfrac{i}{4}$, and $n=2$ of the following hierarchy

$$Q_{1t}=\beta_n\int_{\mathbb{R}} dy_2\Phi^n_{12}Q^-_{12}\sigma. \tag{1.11}$$

The DS equation is also a bi-Hamiltonian system with respect to the two Hamiltonian operators $\Theta^{(1)}_{12}=\sigma$ and $\Theta^{(2)}_{12}=\Phi_{12}\sigma$ defined on off-diagonal matrices. It also possesses two infinite hierarchies of time independent commuting symmetries and constants of motion.

In more detail, this paper is organized as follows: In Sect. 2 we review the algebraic properties of equations in $1+1$. The KdV equation is used as an illustrative example. This is in a sense a summary of [7, 8, 24] although we follow the notation of [7]. In Sect. 3 we derive algorithmically the recursion operators (1.2), (1.9). This derivation is simpler than the one given in [29]; we now use expansions in terms of $d^\ell\delta(y_1-y_2)/dy_1^\ell$, where δ denotes Dirac's function, as opposed to expansions in terms of λ^ℓ. In Sect. 4 we show how Φ_{12} generates extended symmetries σ_{12} and extended gradients of conserved quantities γ_{12}. We then show that σ_{11}, γ_{11} are symmetries and gradients of conserved quantities, respectively. Furthermore, the remarkably rich theory associated with the bi-Hamiltonian factorization of Φ_{12} is developed in this section. In developing this theory we use two important notions: a) The role of Frechét derivative is now played by an appropriate directional derivative, which is naturally motivated from the underlying isospectral problem. b) An extended symmetry σ_{12} can be written

as $\hat{\sigma}_{12} \cdot 1$, where $\hat{\sigma}_{12}$ is an appropriate operator. The Lie algebra of these operators is closed provided they act on appropriate functions H_{12}. Thus in $2+1$ one is dealing with a Lie algebra of operators as opposed to a Lie algebra of functions. In Sect. 5 we give concrete illustrations of the notions introduced in Sect. 4.

We note that Fuchssteiner and one of the authors (ASF) introduced an alternative way for generating symmetries, the so-called mastersymmetry approach. In particular, it is shown in [31] that for the Benjamin-Ono equation $u_t = K$, the map $[\cdot, \tau]_L$, where the bracket $[\,,\,]_L$ is defined in (2.16b), $\tau = xK + u^2 + \frac{3}{2}Hu_x$, and H denotes the Hilbert transform, maps symmetries into symmetries. This approach has been applied to KP in [32], and its general theory has been developed in [33] (for other applications see [34]). However, the τ has certain disadvantages: a) The relationship between τ and the eigenvalue problem has not been established. b) τ is not hereditary. c) It is not known if τ can be used to obtain the second Hamiltonian. In [35] we develop further the theory presented here. In particular, we: i) analyze further the Lie algebra of the starting symmetries and use Φ_{12} to generate time-dependent symmetries, ii) use an isomorphism between Lie and Poisson brackets to show that all these symmetries correspond to extended gradients and hence give rise to conserved quantities, iii) show that the τ mentioned above comes from a time dependent symmetry, and since it corresponds to a gradient cannot be used to generate Φ_{12}, iv) find a non-gradient mastersymmetry (for KP it is $\Phi_{12}^{(2)}\delta_{12}$) which can be used to generate Φ_{12}, v) motivate and verify some of the results presented here and in [35] by establishing that equations in $2+1$ are exact reductions of certain nonlocal evolution equations, of which the algebraic properties are straightforward.

Since two central aspects of integrable equations in $2+1$, namely the IST method and the associated algebraic properties, have now successfully been studied, we speculate that essentially all aspects of equations in $1+1$ will be successfully studied for equations $2+1$. (For example, asymptotics and action-angle formulation of KP have been studied in [36].)

2. Review of Algebraic Properties in $1+1$

We consider evolution equations of the form

$$q_t = K(q), \tag{2.1}$$

where q is an element of some space S of functions on the real line vanishing rapidly for $|x| \to \infty$, and K is some differentiable map on this space depending on q, and on derivatives of q with respect to x. We use the KdV equation as an illustrative example:

$$q_t = q_{xxx} + 6qq_x. \tag{2.2}$$

Equation (2.2) admits the following four-parameter Lie-group of transformations

$$x' = e^{\zeta}(x + \alpha + \gamma t), \quad t' = e^{3\zeta}(t + \beta), \quad q' = e^{-2\zeta}\left(q + \frac{\gamma}{6}\right).$$

The above transformations (space and time translations, Galilean and scaling transformations) are uniquely characterized by the following infinitesimal generators of *symmetries* [37]:

$$\sigma_1 = q_x, \quad \sigma_2 = q_{xxx} + 6qq_x, \quad \Sigma_1 = 1 + 6tq_x, \quad \Sigma_2 = 2q + xq_x + 3t(q_{xxx} + 6qq_x). \tag{23}$$

Actually, the KdV possesses infinitely many symmetries

$$\sigma_n = \Phi^n \sigma_1, \quad \Sigma_n = \Phi^n \Sigma_1, \quad n = 1, 2, \ldots, \tag{2.4}$$

where Φ, the *recursion operator* (a strong symmetry) of the KdV, is given by

$$\Phi = D^2 + 2q + 2DqD^{-1}, \quad (D^{-1}f)(x) \doteqdot \int_{-\infty}^{x} f(\xi)d\xi. \tag{2.5}$$

It turns out that Φ has a certain algebraic property, called *hereditary*, which implies that σ_i, σ_j commute. KdV also possess infinitely many constants of motion; the first few are.

$$I = \int_{-\infty}^{\infty} \varrho_n dx, \quad \varrho_0 = q, \quad \varrho_1 = \frac{q^2}{2}, \quad \varrho_2 = -\frac{q_x^2}{2} + q^3. \tag{2.6a}$$

It is more convenient to work with the gradients of constants of motion:

$$\langle \operatorname{grad} I, v \rangle = \frac{\partial}{\partial \varepsilon} I(q + \varepsilon v)\bigg|_{\varepsilon = 0}, \quad \text{where} \quad \langle f, v \rangle = \int_{-\infty}^{\infty} fv dx$$

is an appropriate scalar product. The functionals I_1, I_2 imply

$$\gamma_1 = q, \quad \gamma_2 = q_{xx} + 3q^2. \tag{2.6b}$$

Equations (2.3), (2.6b) suggest that $\sigma = D\gamma$, i.e. D is a *Noether* operator for the KdV (it relates symmetries to constants of motion). This follows from the fact that KdV is a Hamiltonian, actually a bi-Hamiltonian, system:

$$q_t = D \operatorname{grad} \int_{-\infty}^{\infty} \left(-\frac{q_x^2}{2} + q^3 \right) dx = (D^3 + 2qD + 2Dq) \operatorname{grad} \int_{-\infty}^{\infty} \frac{q^2}{2} dx. \tag{2.7}$$

The two Poisson brackets associated with the above are

$$\begin{gathered} \{I_i, I_j\} = \langle \operatorname{grad} I_i, \Theta_\ell \operatorname{grad} I_j \rangle, \quad \ell = 1 \text{ or } 2, \\ \Theta_1 = D, \quad \Theta_2 = D^3 + 2qD + 2Dq. \end{gathered} \tag{2.8}$$

It can be verified that $\{,\}$ is skew symmetric and satisfies the Jacobi identity.

The notion of a *conserved covariant* γ is a mathematical generalization of the gradient of a conserved quantity. Namely, if the functional I is conserved with respect to a given evolution, then $\gamma = \operatorname{grad} I$ is a conserved covariant. Conversely, if γ is a conserved covariant and if γ is a gradient function, then its *potential* I is a conserved quantity. For example Σ_1 implies a conserved covariant $\Gamma_1 = x - 6tq$ which is a gradient function, hence it implies a conserved quantity $I = \int_{-\infty}^{\infty} (xq - 3tq^2)dx$. However, Γ_2, corresponding to Σ_2, is not a gradient and hence does not correspond to a usual conservation law.

The above discussion motivates the following definitions:

Definition 2.1. (i) A function σ is a symmetry of (2.1) iff

$$\sigma'[K]-K'(\sigma)=0, \tag{2.9}$$

where prime denotes Frechét derivative, i.e.

$$\sigma'[v] \doteq \frac{\partial}{\partial\varepsilon}\sigma(q+\varepsilon v)\bigg|_{\varepsilon=0}. \tag{2.10}$$

(ii) A function γ is a conserved covariant of (2.1) iff

$$\gamma'[K]+K'^{+}[\gamma]=0, \tag{2.11}$$

where K'^{+} is the adjoint of K', namely, $\langle K'^{+}f, g\rangle = \langle f, K'g\rangle$.

(iii) An operator valued function Φ is a recursion operator (strong symmetry) for (2.1) iff

$$\Phi'[K]-[K',\Phi]=0, \tag{2.12}$$

where [,] means commutator.

(iv) An operator valued function Θ is called a Noether operator of (2.1) iff

$$\Theta'[K]-\Theta K'^{+}-K'\Theta=0. \tag{2.13}$$

(v) An operator valued function Θ is called a Hamiltonian operator iff it is skew symmetric and it satisfies

$$\langle a, \Theta'[\Theta b]c\rangle + \text{cyclic permutations}=0. \tag{2.14}$$

vi) An operator valued function Φ is called a hereditary operator iff

$$\Phi'[\Phi v]w-\Phi\Phi'[v]w \quad \text{is symmetric with respect to} \quad v, w. \tag{2.15}$$

(vii) Equation (2.1) is of a Hamiltonian form if it can be written as $q_t=\Theta\gamma$, where Θ is a Hamiltonian operator and γ is a gradient function, i.e. $\gamma'=\gamma'^{+}$.

Proposition 2.1. (i) *If γ is a conserved covariant of* (2.1) *and if γ is a gradient function, then I, the potential of γ, is a conserved quantity for* (2.1).

(ii) *Φ maps σ's to σ's, Φ^{+} maps γ's to γ's, and Θ maps γ's to σ's.*

(iii) *If* (2.1) *is of a Hamiltonian form, then Θ maps γ's to σ's. Furthermore, there is an isomorphism between Lie and Poisson brackets:*

$$[\Theta\gamma_1, \Theta\gamma_2]_L = \Theta\,\mathrm{grad}\langle\gamma_1, \Theta\gamma_2\rangle, \tag{2.16a}$$

where

$$[a,b]_L \doteq a'[b]-b'[a], \tag{2.16b}$$

and γ_1, γ_2 are gradient functions.

(iv) *If Φ is hereditary and Φ is a strong symmetry for σ, then $\Phi^n\sigma_1$, form an abelian algebra.*

(v) *If* (2.1) *is of a bi-Hamiltonian form, then $\Phi=\Theta_2\Theta_1^{-1}$ is a recursion operator of* (2.1).

(vi) *If* (2.1) *is a compatible bi-Hamiltonian system, i.e. if it is bi-Hamiltonian and if $\Theta_1+\Theta_2$ is also a Hamiltonian operator, then $\Phi=\Theta_2\Theta_1^{-1}$ is hereditary. Furthermore, if γ_1 is a conserved gradient of* (2.1), *then $\Phi^{+n}\gamma_1$ are also conserved gradients. Thus* (2.1) *possesses infinitely many commuting symmetries and infinitely many conserved quantities in involution.*

Given the isospectral eigenvalue problem associated with (2.1) there is an algorithmic way of obtaining Φ. Furthermore, if Φ has a complete set of eigenfunctions it must be hereditary:

Proposition 2.2. *Let*

$$V_x=U(q,\lambda)V \tag{2.17}$$

be a linear isospectral eigenvalue problem associated with (2.1). *Let G_λ denote the gradient of the eigenvalue λ. If G_λ satisfies*

$$\Psi G_\lambda=\mu(\lambda)G_\lambda, \tag{2.18}$$

then $\Phi=\Psi^+$ is a hereditary operator (provided G_λ form a complete set).

3. Derivation of Recursion Operators

A. The Schrödinger Eigenvalue Problem

Proposition 3.1. *The Schrödinger equation* (1.1) *is associated with the following equation:*

$$\delta_{12}q_{1_t}=D\Psi_{12}T_{12}-2q_{12}^-a_{12}, \tag{3.1}$$

where $q_{12}^{\pm}$ are given by (1.2b), *δ denotes the Dirac delta function, T, a are arbitrary functions of the arguments indicated,*

$$\delta_{12}\doteq\delta(y_1-y_2),\quad T_{12}\doteq T(x,y_1,y_2),\quad a_{12}\doteq a(y_1,y_2), \tag{3.2}$$

and Ψ_{12} is given by

$$\Psi_{12}\doteq D^2+q_{12}^++D^{-1}q_{12}^+D+D^{-1}q_{12}^-D^{-1}q_{12}^-. \tag{3.3}$$

To derive the above result first write Eq. (1.1) in matrix form

$$W_x=UW,\quad W\doteq\begin{pmatrix} w\\ w_x\end{pmatrix},\quad U=\begin{pmatrix}0 & 1\\ -q-\alpha D_y & 0\end{pmatrix}. \tag{3.4}$$

Equation (3.4) is compatible with

$$W_t=VW,\quad V\doteq\begin{pmatrix}A & 2C\\ B & E\end{pmatrix} \tag{3.5}$$

if

$$U_t=V_x-[U,V]. \tag{3.6}$$

The operator equation (3.6) implies

$$A_x = B + 2C\hat{q}, \quad E_x = -B - 2\hat{q}C, \quad 2C_x = E - A,$$
$$q_t = -B_x - \hat{q}A + E\hat{q}, \quad \hat{q} \doteq q + \alpha D_y. \tag{3.7}$$

The above equations yield

$$A = -C_x + D^{-1}[C, \hat{q}] + A_0, \quad A_{0x} = 0,$$
$$B = -C_{xx} - [C, q]^+, \tag{3.8}$$

$$E = C_x + D^{-1}[C, \hat{q}] + A_0,$$
$$q_t = C_{xxx} + [\hat{q}, C]_x^+ + [\hat{q}, C_x]^+ + [\hat{q}, D^{-1}[\hat{q}, C]] + A_0\hat{q} - \hat{q}A_0, \tag{3.9}$$

where $[\,,]^+$ is the usual anticommutator of two operators. We represent the operator C by:

$$(Cf)(x, y_1) = \int_{\mathbb{R}} dy_2 T(x, y_1, y_2) f(x, y_2), \tag{3.10}$$

similarly,

$$A_0 f_1 = 2\int_{\mathbb{R}} dy_2 a_{12} f_2 .$$

Then

$$(\hat{q}_1 C \pm C\hat{q}_1) f_1 = \int_{\mathbb{R}} dy_2 (q_{12}^{\pm} T_{12}) f_2 ,$$
$$[\hat{q}_1, D^{-1}[\hat{q}_1, C]] f_1 = \int_{\mathbb{R}} dy_2 (q_{12}^{-} D^{-1} q_{12}^{-} T_{12}) f_2 , \tag{3.11}$$
$$(A_0\hat{q}_1 - \hat{q}_1 A_0) f_1 = -\int_{\mathbb{R}} dy_2 2q_{12}^{-} a_{12} f_2 .$$

Hence applying the arbitrary function f to the operator equation (3.9) we obtain

$$\delta_{12} q_{2_t} = T_{12_{xxx}} + (q_{12}^{+} T_{12})_x + q_{12}^{+} T_{12_x} + q_{12}^{-} D^{-1} q_{12}^{-} T_{12} - 2q_{12}^{-} a_{12} . \tag{3.12}$$

Remark 3.1. It is easily verified that the following important commutator operator relationships are valid:

$$[q_{12}^{-}, h_{12}] = 0, \quad [q_{12}^{+}, h_{12}] = 2\alpha h'_{12}, \quad [\Psi_{12}, h_{12}] = 4\alpha h'_{12}; \tag{3.13}$$

hereafter h_{12} is any arbitrary function $h(y_1 - y_2)$ and h'_{12} denotes its derivative with respect to y_1.

Proposition 3.1 can be used to derive nonlinear evolution equations related to (1.1). One needs only to assume appropriate expansions of T_{12}, a_{12}. We give two examples:

Example 1.

$$T_{12} = \sum_{j=0}^{n} \delta_{12}^{j} T_{12}^{(j)}, \quad T_{12}^{(n)} = C_n, \quad a_{12} = 0, \tag{3.14}$$

where $\delta_{12}^{j} \doteq \partial^j \delta_{12} / \partial y_1^j$, C_n an arbitrary constant. Then

$$q_{1_t} = \beta_n \int_{\mathbb{R}} dy_2 \delta_{12} D \Psi_{12}^{n+1} \cdot 1, \quad n = 1, 2, \ldots . \tag{3.15}$$

To derive (3.15), use Eqs. (3.14) in (3.12) and use (3.13c) with $h_{12}=\delta_{12}$.

$$\delta_{12}q_{2_t}=D\left(\sum_{j=0}^{n}\delta_{12}^{j}\Psi_{12}T_{12}^{(j)}+4\alpha\sum_{j=1}^{n+1}\delta_{12}^{j}T_{12}^{(j-1)}\right).$$

Equating the coefficients of δ_{12}^{n+1} and δ_{12}^{j}, $1\leqq j\leqq n$ to zero, we obtain

$$T_{12_x}^{(n)}=0,\qquad T_{12}^{(j-1)}=-\frac{1}{4\alpha}\Psi_{12}T_{12}^{(j)}.$$

Hence

$$T_{12}^{(n-j)}=\left(-\frac{1}{4\alpha}\right)^{j}C_n\Psi_{12}^{j}\cdot 1,\qquad \delta_{12}q_{2_t}=\delta_{12}D\Psi_{12}T_{12}^{(0)}=\left(-\frac{1}{4\alpha}\right)^{n}C_n\delta_{12}D\Psi_{12}^{n+1}\cdot 1.$$

Thus (3.15) follows with the normalization $(-1)^n\beta_n=(4\alpha)^{-n}C_n$.

Example 2.

$$T_{12}=\sum_{j=0}^{n}\delta_{12}^{j}T_{12}^{(j)},\qquad T_{12}^{(n)}=0,\qquad a_{12}=-\frac{1}{2}C_n\delta_{12}^{n}. \tag{3.16}$$

Then

$$q_{1_t}=\beta_n\int_{\mathbf{R}}dy_2\delta_{12}D\Psi_{12}^{n}D^{-1}q_{12}^{-}\cdot 1,\qquad n=1,2,\ldots, \tag{3.17}$$

with the normalization $C_n=(-1)^n(4\alpha)^n\beta_n$.

Remark 3.2. 1. The operators Φ_{12}, Ψ_{12} defined by (1.2) and (3.3), respectively, are related via

$$\Phi_{12}D=D\Psi_{12}. \tag{3.18}$$

Hence the hierarchy of Eqs. (3.15) can be written as

$$q_{1_t}=\beta_n\int_{\mathbf{R}}dy_2\delta_{12}D\Psi_{12}^{n+1}\cdot 1=\beta_n\int_{\mathbf{R}}dy_2\delta_{12}\Phi_{12}^{n}(\Phi_{12}D)\cdot 1. \tag{3.19}$$

The KP equation corresponds to $n=1$ and $\beta_1=\frac{1}{2}$; the next equation of the class (for $\beta_2=\frac{1}{2}$) is

$$\begin{aligned}q_t&=q_{xxxxx}+10qq_{xxx}+20q_xq_{xx}+30q^2q_x\\&\quad+5\alpha^2(2q_{yyx}+D^{-1}(q^2)_{yy}+2q_xD^{-2}q_{yy}+4q_yD^{-1}q_y+4qD^{-1}q_{yy})+5\alpha^4D^{-3}q_{yyyy}.\end{aligned}$$

2. Similarly, the hierarchy of Eqs. (3.17) can be written as

$$q_{1_t}=\beta_n\int_{\mathbf{R}}dy\delta_{12}D\Psi_{12}^{n}(D^{-1}q_{12}^{-}\cdot 1)=\beta_n\int_{\mathbf{R}}dy_2\delta_{12}\Phi_{12}^{n}q_{12}^{-}\cdot 1. \tag{3.20}$$

For $n=1$ and $\beta_1=\frac{1}{4}$ the above becomes $q_{1_t}=\alpha q_{1_{y_1}}$, i.e. it corresponds to a y-translation.

B. The 2×2 AKNS in 2+1

Proposition 3.2. *Equation* (1.8) *is associated with the following equation:*

$$\delta_{12}Q_{2_t}=\sigma\Psi_{12}V_{12_0}, \tag{3.21}$$

where V_{12_O} denotes an arbitrary off-diagonal matrix and the operator Ψ_{12} (acting only on off-diagonal matrices) is given by

$$\Psi_{12} \doteq \sigma(P_{12} - Q_{12}^- P_{12}^{-1} Q_{12}^-), \quad P_{12}F_{12} \doteq F_{12_x} - JF_{12_{y_1}} - F_{12_{y_2}}J. \tag{3.22}$$

To derive the above note that (1.8) can be written as

$$W_x = \hat{Q}W, \quad \hat{Q} = Q + JD_y. \tag{3.23}$$

Equation (3.23) is compatible with $W_t = \hat{V}W$ if

$$\hat{Q}_t = \hat{V}_x - [\hat{Q}, \hat{V}]. \tag{3.24}$$

We represent the operator $\hat{V}$ by

$$(\hat{V}F)(x, y_1) \doteq \int_{\mathbb{R}} dy_2 V(x, y_1, y_2)F(x, y_2). \tag{3.25}$$

Then $[\hat{Q}, \hat{V}] = \int_{\mathbb{R}} dy_2(\hat{Q}_{12}V_{12})F_2$, where $\hat{Q}_{12}F_{12} \doteq Q_1F_{12} - F_{12}Q_2 + JF_{12_{y_1}} + F_{12_{y_2}}J$. Hence (3.24) implies $\delta_{12}Q_{1_t} = (D - \hat{Q}_{12})V_{12}$. Splitting this equation into diagonal and off-diagonal parts we obtain

$$\delta_{12}Q_{2_t} = P_{12}V_{12_O} - Q_{12}^- V_{12_D}, \quad P_{12}V_{12_D} - Q_{12}^- V_{12_O} = 0. \tag{3.26}$$

where V_{12_D} and V_{12_O} are the diagonal and off-diagonal parts of V_{12}. Hence Eq. (3.21) follows.

Remark 3.3. The operator Ψ_{12} satisfies the following important commutator relationship:

$$[\Psi_{12}, h_{12}]F_{12_O} = -2\alpha h'_{12}F_{12_O}, \tag{3.27}$$

where F_{12_O} is the off-diagonal part of the arbitrary matrix function F_{12} and prime denotes differentiation with respect to y_1.

The above relationship follows by considering the diagonal and off-diagonal parts of the following equation

$$[D - \hat{Q}_{12}, h_{12}]F_{12} = -2\alpha h'_{12}\sigma F_{12_O}. \tag{3.28}$$

Remark 3.4. Assuming

$$V_{12_O} = \sum_{j=0}^{n} \delta_{12}^j v_{12}^{(j)}, \quad v_{12}^{(j)} \text{ off-diagonal}, \tag{3.29}$$

Eq. (3.21) implies

$$Q_{1_t} = \sigma \int_{\mathbb{R}} dy_2 \delta_{12} \Psi_{12}^n Q_{12}^- v_{12_D}; \quad P_{12}v_{12_D} = 0, \tag{3.30}$$

where v_{12_D} is any diagonal matrix solving (3.30b).

To derive (3.30) note that Eqs. (3.21) and (3.27) imply

$$\delta_{12}Q_{2_t} = \sigma\left(\sum_{j=0}^{n} \delta_{12}^j \Psi_{12} v_{12}^{(j)} - 2\alpha \sum_{j=1}^{n+1} \delta_{12}^j v_{12}^{(j-1)}\right). \tag{3.31}$$

Equating the coefficients of $\delta_{12}^{n+1}, \delta_{12}^j$, $n \geqq j \geqq 1$, to zero we obtain

$$v_{12}^{(n)} = 0, \quad v_{12}^{(0)} = \frac{1}{(2\alpha)^{n-1}} \Psi_{12}^{n-1} v_{12}^{(n-1)}, \quad 2\alpha v_{12}^{(n-1)} = \Psi_{12} v_{12}^{(n)}. \tag{3.32}$$

Equation (3.32c) can be written as

$$2\alpha\sigma v_{12}^{(n-1)}=P_{12}v_{12}^{(n)}-Q_{12}^{-}v_{12_D}, \quad 0=P_{12}v_{12_D}-Q_{12}^{-}v_{12}^{(n)}, \tag{3.33}$$

where v_{12_D} is an arbitrary diagonal matrix. Hence (3.32c) and (3.32a) imply $v_{12}^{(n-1)}=\left(\frac{-1}{2\alpha}\right)\sigma Q_{12}v_{12_D}$, where v_{12_D} solves $P_{12}v_{12_D}=0$. Hence $v_{12}^{(0)}=-1/(2\alpha)^n\Psi_{12}^{n-1}\sigma Q_{12}^{-}v_{12_D}$ and the coefficient δ_{12}^{0} imply (3.30).

Remark 3.5. Let Φ_{12} be defined by (1.9a), then one easily verifies that

$$\Phi_{12}\sigma=\sigma\Psi_{12}. \tag{3.34}$$

Equation (3.30), for special choices of v_{12_D} yields hierarchies of integrable equations:

Example 1. Let $v_{12_D}=\sigma$, then (3.30) implies

$$Q_{1_t}=-\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\sigma\Psi_{12}^nQ_{12}^{+}I=\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\Phi_{12}^nQ_{12}^{-}\sigma. \tag{3.35}$$

To derive (3.35) note that $Q_{12}^{-}\sigma=-\sigma Q_{12}^{+}$. Also (3.34) implies that $\Phi_{12}^n\sigma=\sigma\Psi_{12}^n$. Hence the integral of Eq. (3.30) implies

$$-\sigma\Psi_{12}^nQ_{12}^{+}I=-\Phi_{12}^n\sigma Q_{12}^{+}I=\Phi_{12}^nQ_{12}^{-}\sigma.$$

Remark 3.6. Equations (3.35) for $n=0,1,2$ become

$$Q_t=\sigma Q, \quad \beta_0=-\tfrac{1}{2}, \tag{3.36a}$$

$$Q_t=Q_x, \quad \beta_1=-\tfrac{1}{2}, \tag{3.36b}$$

$$\left.\begin{aligned}Q_t&=-\beta_2[2\sigma(Q_{xx}+\alpha^2Q_{yy})-QA+AQ]\\(D_x-JD_y)A&=-2(D_x+JD_y)\sigma Q^2\end{aligned}\right\}. \tag{3.36c}$$

Equations (3.36c) under the reduction $q_2=\bar{q}_1=q$ yield the DS equation $\left(\beta_2=-\frac{i}{4}\right)$

$$\begin{aligned}iq_t+\tfrac{1}{2}(q_{xx}+\alpha^2q_{yy})&=q(\phi-|q|^2),\\ \phi_{xx}-\alpha^2\phi_{yy}&=2|q|_{xx}^2.\end{aligned} \tag{3.37}$$

Example 2. Let $v_{12_D}=I$, then (3.30) implies

$$Q_{1_t}=-\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\sigma\Psi_{12}^nQ_{12}^{+}=\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\Phi_{12}^nQ_{12}^{-}I. \tag{3.38}$$

Equations (3.38) for $n=0,1,2$ become

$$Q_t=0, \tag{3.39a}$$

$$Q_t=\alpha Q_y, \quad \beta_1=-\tfrac{1}{2}, \tag{3.39b}$$

$$\left.\begin{aligned}Q_t&=\beta_2[-4\alpha\sigma Q_{xy}+BQ-QB]\\(D_x-JD_1)B&=4\alpha\sigma(Q_1^2)_y\end{aligned}\right\}. \tag{3.39c}$$

Equations (3.39c) under the reduction $q_2=\bar{q}_1=\bar{q}$ yield ($\beta_2=-\frac{1}{4}$)

$$q_t=\alpha q_{xy}+uq\,,\qquad u_{xx}-\alpha^2u_{yy}=2\alpha|q|^2_{xy}\,. \tag{3.39d}$$

C. Motivation

A crucial step in deriving the recursion operator associated with the Schrödinger equation was to use an integral representation of the operator C [see Eq. (3.10)]. Also in deriving the theory for recursion operators we will need an appropriate Frechét derivative. Both, the integral representation (3.10) and the above Frechét derivative can be motivated as follows:

Consider

$$w_{xx}+\tilde{q}w+\alpha w_y=0\,;\qquad (\tilde{q}f)(x,y)=\int_{\mathbb{R}} dy_2q(x,y,y_2)f(x,y_2)\,. \tag{3.40}$$

Equation (1.1) can be thought of as the reduction of (3.40) under $q(x,y_1,y_2)=\delta_{12}q(x,y_1)$. It is clear that $\tilde{q}$ satisfies an equation similar to (3.9) where q is replaced by $\tilde{q}$. Since the operator $\tilde{q}$ has the integral representation (3.40b), one is lead to consider a similar integral representation for the operator C [Eq. (3.10)]. An equation similar to (3.12) is also valid for $\tilde{q}$, where $q^{\pm}_{12}$ are replaced by $\tilde{q}^{\pm}_{12}$,

$$\tilde{q}^{\pm}_{12}f_{12}\doteq\int_{\mathbb{R}} dy_3(q_{13}f_{32}\pm f_{13}q_{32})+\alpha(D_1\mp D_2)f_{12}\,. \tag{3.41}$$

The Frechét derivative of $\tilde{q}^{\pm}_{12}f_{12}$ in the direction σ_{12} yields

$$\tilde{q}^{\pm\prime}_{12}[\sigma_{12}]f_{12}\doteq\int_{\mathbb{R}} dy_3(\sigma_{13}f_{32}\pm f_{13}\sigma_{32})\,. \tag{3.42}$$

This is precisely the directional derivative we use in Sect. 4. More details on the concept of equations in 2 + 1 dimensions as exact reductions of nonlocal evolution equations are presented in [35, Sect. V].

4. Algebraic Properties in 2+1

The theory of algebraic properties in 2 + 1 is based on the following concepts: a) A crucial step in deriving the recursion operator associated with a given two-dimensional eigenvalue problem is the use of an integral representation of operators depending on q and $\partial/\partial y$. In KP for example $\hat{q}\doteq q+\alpha\partial/\partial y$ is represented by

$$(q_1+\alpha D_1)f_{12}\doteq\int_{\mathbb{R}} dy_3q_{13}f_{32}\,. \tag{4.1a}$$

The above mapping between an operator and its kernel induces a mapping between derivatives:

$$\hat{q}_{1_d}[\sigma_{12}]f_{12}=\int_{\mathbb{R}} dy_3\sigma_{13}f_{32}\,, \tag{4.1b}$$

where $\hat{q}_{1_d}[\sigma_{12}]$ denotes the directional derivative of the operator valued function $\hat{q}_1$ in the direction σ_{12}. Using an appropriate bilinear form [see (4.7)–(4.8)] Eqs. (4.1) imply

$$\hat{q}^*_2f_{12}=(q_2-\alpha D_2)f_{12}=\int_{\mathbb{R}} dy_3f_{13}q_{32}\,,\qquad \hat{q}^*_{1_d}[\sigma_{12}]f_{12}=\int_{\mathbb{R}} dy_3f_{13}\sigma_{32}\,. \tag{4.2}$$

The recursion operator Φ_{12} depends only on $\hat{q}_1$ and $\hat{q}_1^*$, thus one is able to define $\Phi_{12_d}[\sigma_{12}]$. b) The theory of symmetries for equations in $1+1$ is based on the existence of "starting" symmetries K^0, which via Φ generate infinitely many symmetries. For example, for the KdV $K^0=q_x$. For equations in $2+1$ we find that the starting symmetries K_{12}^0 can be written as $\hat{K}_{12}^0 H_{12}$, where $\hat{K}_{12}^0$ is an operator and H_{12} is a suitable function [for the KP $H_{12}=H_{12}(y_1,y_2)$]. The operators $\hat{K}_{12}^0$ depend only on $\hat{q}_1, \hat{q}_1^*$ and thus $\hat{K}_{12_d}^0$ is well defined. The Lie algebra of the starting operators $\hat{K}_{12}^0$ acting on H_{12} is closed. This fact, which is of fundamental importance for the theory developed both here and in [35], can also be traced back to the integral representation of the fundamental operator $\hat{q}$. For example, Eq. (4.1b) implies:

$$\hat{q}_{1_d}[\sigma_{12}]f_{12}-\hat{q}_{1_d}[f_{12}]\sigma_{12}=\int_{\mathbb{R}} dy_3(\sigma_{13}f_{32}-f_{13}\sigma_{32}).$$

Also using

$$\hat{q}_{1_d}[\hat{q}_1\sigma_{12}]f_{12}=\int_{\mathbb{R}} dy_3(\hat{q}_1\sigma_{12})_{13}f_{32}=\int_{\mathbb{R}^2} dy_3 dy_3' f_{32}q_{13'}\sigma_{3'3},$$

it follows that

$$\hat{q}_{1_d}[\hat{q}_1\sigma_{12}]f_{12}-\hat{q}_{1_d}[\hat{q}_1 f_{12}]\sigma_{12}=\hat{q}_1\int_{\mathbb{R}} dy_3(\sigma_{13}f_{32}-f_{13}\sigma_{32}).$$

The above equation can be written as

$$[\hat{q}_1 f_{12}, \hat{q}_1\sigma_{12}]_d=\hat{q}_1[\sigma_{12}, f_{12}]_I,$$

where the following brackets have been motivated from the above example:

$$[\hat{K}_{12}^{(1)}H_{12}^{(1)}, \hat{K}_{12}^{(2)}H_{12}^{(2)}]_d \doteq K_{12_d}^{(1)}[\hat{K}_{12}^{(2)}H_{12}^{(2)}]H_{12}^{(1)}-\hat{K}_{12_d}^{(2)}[\hat{K}_{12}^{(1)}H_{12}^{(1)}]H_{12}^{(2)}, \tag{4.3a}$$

$$[H_{12}^{(1)}, H_{12}^{(2)}]_I \doteq \int_{\mathbb{R}} dy_3(H_{13}^{(1)}H_{32}^{(2)}-H_{13}^{(2)}H_{32}^{(1)}). \tag{4.3b}$$

In $1+1$, one considers the Lie algebra of *functions*; in $2+1$ one, instead, considers the Lie algebra of *operators*, thus equations in $2+1$ have richer algebraic structure than equations in $1+1$. c) The recursion operator Φ_{12} and the starting operators $\hat{K}_{12}^0$ have simple commutator relations with δ_{12} or more generally with $h_{12}=h(y_1-y_2)$.

Notation. We will consider exactly solvable evolution equations of the form $q_t=K(q)$, where $q(x,y,t)$ is an element of a suitable space S of functions vanishing rapidly for large x, y. Let K be a differentiable map on this space (we assume for convenience that it does not depend explicitly on x, y, t). The above equation is a member of a hierarchy generated by Φ_{12}, hence more generally, we shall study $q_t=K^{(n)}(q)$. Fundamental in our theory is to write these equations in the form

$$q_{1_t}=\int_{\mathbb{R}} dy_2\delta_{12}\Phi_{12}^n\hat{K}_{12}^0\cdot 1 \doteq \int_{\mathbb{R}} dy_2\delta_{12}K_{12}^{(n)}=K_{11}^{(n)} \tag{4.4$_n$}$$

(in the matrix case, 1 is replaced by the identity matrix I), where $K_{12}^{(n)}(q_1,q_2)$ belong to a suitably extended space $\tilde{S}$, and Φ_{12}, $\hat{K}_{12}^0$ are operator valued functions in $\tilde{S}$. For an arbitrary function $K_{12}(q_1,q_2)$ we define the total *Frechét* derivative by

$$K_{12_f}[F] \doteq K_{12_{q_1}}[F_{11}]+K_{12_{q_2}}[F_{22}], \tag{4.5a}$$

where $K_{12_{q_i}}$ denotes the Frechét derivative of K_{12} with respect to q_i, i.e.

$$K_{12_{q_i}}[F_{ii}] \doteq \frac{\partial}{\partial\varepsilon} K_{12}(q_i+F_{ii}, q_j)\bigg|_{\varepsilon=0}, \quad i,j=1,2, \quad i \neq j. \tag{4.5b}$$

We also define a special *directional* derivative, dictated by the underlying isospectral problem and denoted by K_{12_d}. This derivative is linear, satisfies the Leibnitz rule and is related to the above Frechét derivative by

$$K_{12_d}[\delta_{12}F_{12}] = K_{12_f}[F]. \tag{4.6}$$

For arbitrary functions $f_{12} \in \tilde{S}$ and $g_{12} \in \tilde{S}^*$, where S^* denotes the dual of S, we define the following symmetric *bilinear form*

$$\langle g_{12}, f_{12}\rangle \doteq \int_{\mathbb{R}^3} dx dy_1 dy_2 \operatorname{trace} g_{21} f_{12}, \quad f_{12}, g_{12} \text{ matrices}, \tag{4.7}$$

where obviously the trace is dropped if f_{12}, g_{12} are scalars. The operator L_{12}^* is called the adjoint of L_{12} with respect to the above bilinear form, iff

$$\langle L_{12}^* g_{12}, f_{12}\rangle = \langle g_{12}, L_{12} f_{12}\rangle. \tag{4.8}$$

For arbitrary functions $f \in S$ and $g \in S^*$, we define the following symmetric *bilinear form*

$$(g, f) \doteq \int_{\mathbb{R}^2} dx dy \operatorname{trace} g f, \quad f, g \text{ matrices}. \tag{4.9}$$

The operator L^+ is called the adjoint of L with respect to the bilinear form (4.9) iff

$$(L^+ g, f) = (g, Lf). \tag{4.10}$$

Remark 4.1. Definitions (4.7) and (4.9) imply

$$\langle \delta_{12} g_{12}, f_{12}\rangle = \langle g_{12}, \delta_{12} f_{12}\rangle = (g_{11}, f_{11}). \tag{4.11}$$

Let I be a functional given by

$$I = \int_{\mathbb{R}^2} dx dy_1 \operatorname{trace} \varrho_{11} = \int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} \operatorname{trace} \varrho_{12}, \quad \varrho_{12} = \varrho(x, y_1, y_2, t) \in \tilde{S} \tag{4.12}$$

(if ϱ_{12} is a scalar, then omit trace).

The *extended gradient* $\operatorname{grad}_{12} I$ of this functional is defined by

$$\langle \operatorname{grad}_{12} I, \cdot\rangle \doteq I_d[\cdot] = \int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} \varrho_{12_d}[\cdot]. \tag{4.13}$$

The gradient of I, $\operatorname{grad} I$, is instead defined by

$$(\operatorname{grad} I, \cdot) \doteq I_f[\cdot] = \int_{\mathbb{R}^2} dx dy \varrho_f[\cdot]. \tag{4.14}$$

It is easily seen that a function $\gamma_{12} \in \tilde{S}^*$ is an *extended gradient* function (i.e. it has a *potential* I) iff

$$\gamma_{12_d} = \gamma_{12_d}^*. \tag{4.15a}$$

A function $\gamma \in S$ is a gradient function iff

$$\gamma_f = \gamma_f^+. \tag{4.15b}$$

Some of the above notions make sense only if for certain functions the directional derivative exists. Such functions are called admissible.

Throughout this paper m, n denote non-negative integers.

A. Basic Notions

Definition 4.1. i) An operator valued function L_{12} is called *admissible* if its directional derivative is well defined.

ii) A function K_{12} is called admissible if it can be written as $K_{12}=\hat{K}_{12}H_{12}$, where $\hat{K}_{12}$ is an admissible operator and H_{12} is an appropriate function [for KP, $H_{12}=H_{12}(y_1, y_2)$].

In analogy with Sect. 2 we give the following definitions:

Definition 4.2. Consider the evolution equation

$$q_{1_t}=\int_{\mathbb{R}} dy_2 \delta_{12} K_{12}=K_{11}. \tag{4.16}$$

i) The function σ_{12} is called an *extended symmetry* of (4.16) iff

$$\sigma_{12_t}[K]=(\delta_{12}K_{12})_d[\sigma_{12}]. \tag{4.17}$$

ii) The function γ_{12} is called an extended *conserved covariant* of (4.16) iff

$$\gamma_{12_t}[K]+(\delta_{12}K_{12})_d^*[\gamma_{12}]=0. \tag{4.18}$$

iii) The admissible operator valued function Φ_{12} is called a *strong symmetry* (recursion operator) of (4.16) iff

$$\Phi_{12_t}[K]+[\Phi_{12},(\delta_{12}K_{12})_d]=0. \tag{4.19}$$

iv) The admissible operator valued function Θ_{12} is called a *Noether* operator of (4.16) iff

$$\Theta_{12_t}[K]-\Theta_{12}(\delta_{12}K_{12})_d^*-(\delta_{12}K_{12})_d\Theta_{12}=0. \tag{4.20}$$

v) The admissible operator valued function Φ_{12} is called a hereditary operator iff

$$\Phi_{12_d}[\Phi_{12}f_{12}]g_{12}-\Phi_{12}\Phi_{12_d}[f_{12}]g_{12} \quad \text{is symmetric with respect to } f_{12}, g_{12} \tag{4.21}$$

Remark 4.2. i) σ_{12} is an extended symmetry of (4.16) iff σ_{12} commutes with $\delta_{12}K_{12}$,

$$[\sigma_{12},\delta_{12}K_{12}]_d=0. \tag{4.22}$$

This follows from the fact that $\sigma_{12_d}[\delta_{12}K_{12}]=\sigma_{12_t}[K]$.

ii) If in (4.12), ϱ_{12} is an admissible function, $\varrho_{12}=\hat{\varrho}_{12}H_{12}$; then the functional I depends on H_{12}, $I=I(H_{12})$, and $\gamma_{12}\doteq\text{grad}_{12}I$, defined by (4.13), is also an admissible function $\gamma_{12}=\hat{\gamma}_{12}H_{12}$, enjoying the property (4.15a) for every H_{12}. If, for instance, $I=\int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} q_{12}^{+} D^{-1} q_{12}^{-} H_{12}$ and the directional derivative is defined in (4.13) [see also (4.1b) and (4.2)], then $\gamma_{12}=4D^{-1}q_{12}^{-}H_{12}$ is the corresponding extended gradient.

iii) If γ_{12} in addition to satisfying (4.18) is also an extended gradient function, then its potential I is a conserved quantity of (4.16). This follows from the following:

$$I_t = I_f[K] = I_d[\delta_{12}K_{12}] = \langle \gamma_{12}, \delta_{12}K_{12} \rangle ,$$

where $\gamma_{12} = \mathrm{grad}_{12} I$. The derivative of the above in the arbitrary direction v_{12} is zero if (4.18) holds.

iv) Φ_{12} is a strong symmetry for a_{12} iff

$$\Phi_{12_d}[a_{12}] + [\Phi_{12}, a_{12_d}] = 0 . \tag{4.23a}$$

Hence Eq. (4.21) implies that Φ_{12} is a strong symmetry for $(\delta_{12}K_{12})$ (see Lemma 4.1).

v) Θ_{12} is a Noether operator for a_{12} iff

$$\Theta_{12_d}[a_{12}] - \Theta_{12} a^*_{12_d} - a_{12_d}\Theta_{12} = 0 . \tag{4.23b}$$

Hence Eq. (4.20) implies that Θ_{12} is a Noether operator for $(\delta_{12}K_{12})$ (see Lemma 4.1).

vi) In the above definitions we assume that $\sigma_{12}, \gamma_{12}, \Theta_{12}, \Phi_{12}$ do not explicitly depend on t. Otherwise, $\sigma_{12_f}[K]$ should be replaced by $\partial\sigma_{12}/\partial t + \sigma_{12_f}[K]$; similarly, for $\gamma_{12_f}, \Theta_{12_f}, \Phi_{12_f}$.

Remark 4.3. i) Φ_{12} maps solutions of (4.17) to solutions of (4.17);

ii) Φ^*_{12} maps solutions of (4.18) to solutions of (4.18);

iii) Θ_{12} maps solutions of (4.18) to solutions of (4.17);

iv) if Θ_{12} solves (4.20) and Φ_{12} solves (4.19) then $\Phi^n\Theta_{12}$ also solves (4.20).

Definitions 4.2 make sense only if $(\delta_{12}K_{12})_d$ exists. For equations generated by Φ_{12}, $(\delta_{12}K_{12})_d$ is well defined:

Lemma 4.1. *Assume that the admissible operators Φ_{12} and $\mathring{K}^0_{12}$ satisfy the following operator equations*

$$[\Phi_{12}, h_{12}] = -\beta h'_{12} , \tag{4.24a}$$

$$[\mathring{K}^0_{12}, h_{12}] = -\bar{\beta}\mathring{S}_{12}h'_{12} , \tag{4.24b}$$

where $\beta, \bar{\beta}$ are constants, $\mathring{S}_{12}$ is some admissible operator, $h_{12} = h(y_1 - y_2)$ and prime denotes derivative with respect to y_1. Then all notions introduced in Definitions 4.2 are well defined for any Eq. $(4.4)_n$. In particular:

$$(\delta_{12}\Phi^n_{12}\mathring{K}^0_{12} \cdot 1)_d = ((\Phi_{12} + \beta\mathscr{D})^n(\mathring{K}^0_{12} + \bar{\beta}\mathring{S}_{12}\mathscr{D})\delta_{12})_d , \tag{4.25}$$

where the operator $\mathscr{D}$ is defined by

$$[\mathscr{D}, \mathring{a}_{12}] = 0 , \quad \mathscr{D} \cdot h_{12} = h'_{12} , \tag{4.26}$$

and $\mathring{a}_{12}$ is any admissible operator. Thus

$$(\Phi_{12} + \beta\mathscr{D})^n\delta_{12} = \sum_{\ell=0}^{n} \beta^\ell \binom{n}{\ell} \Phi^{n-\ell}_{12}\delta^\ell_{12} , \quad \binom{n}{\ell} \doteq \frac{n!}{(n-\ell)!\,\ell!} . \tag{4.27}$$

Equations (4.24) imply that $\delta_{12}\Phi^n_{12}\mathring{K}^0_{12} \cdot 1 = (\Phi_{12} + \beta\mathscr{D})^n(\mathring{K}^0_{12} + \bar{\beta}\mathring{S}_{12}\mathscr{D})\delta_{12}$ which is an admissible function since Φ_{12}, $\mathring{K}^0_{12}$, $\mathring{S}_{12}$ are admissible operators.

Remark 4.4. i) For the two-dimensional AKNS we use two starting operators $\hat{K}^0_{12}$; both of these operators commute with h_{12} (i.e. $\tilde{\beta}=0$). For the two-dimensional Schrödinger we also use two starting operators $\hat{K}^0_{12}$; one of them commutes with h_{12}, the other implies $\tilde{\beta}=\frac{\beta}{2}$, $\hat{S}_{12}=D$.

ii) It is clear that the theory presented here, suitably modified, is also valid for more general commutator relations than the ones given by (4.24). In investigating a new eigenvalue problem one first computes the commutator of Φ_{12} and $\hat{K}^0_{12}$ with h_{12}; one then builds a general theory based on these commutator relations.

iii) We remark that Eq. (4.24a) could be derived directly from the underlying isospectral problem without using the explicit form of Φ_{12}. As an example, in Sect. 4.E we show that the equation $\Phi^*_{12}W_1W_2^+=4\lambda W_1W_2^+$ (which is a direct consequence of the spectral problem $W_{xx}+\hat{q}W=\lambda W$) implies Eq. (4.24a), with $\beta=-4\alpha$.

The usefulness of the extended symmetries and the extended gradients follows from the fact that their reduction yields symmetries and gradients, respectively.

Theorem 4.1. *Assume that the admissible operators Φ_{12}, $\hat{K}^0_{12}$, satisfy*

$$[\Phi_{12},\delta_{12}]=-\beta\delta'_{12}, \tag{4.28a}$$

$$[\hat{K}^0_{12},\delta_{12}]=-\tilde{\beta}\hat{S}_{12}\delta'_{12}, \tag{4.28b}$$

where $\beta,\tilde{\beta}$ are constants, $\hat{S}_{12}$ is such that

$$\hat{S}_{12_d}[\cdot]H_{12}=\hat{S}_{12_f}[\cdot]H_{12}=0$$

and prime denotes derivative with respect to y_1. Then:

i) *If σ_{12} is an extended symmetry of*

$$q_{1_t}=\int_{\mathbb{R}}dy_2\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1=\int_{\mathbb{R}}dy_2\delta_{12}K^{(n)}_{12}=K^{(n)}_{11}, \tag{4.29}$$

σ_{11} is a symmetry of (4.29).

ii) *Similarly, if γ_{12} is an extended conserved covariant of* (4.29), *γ_{11} is a conserved covariant of* (4.29).

iii) *If γ_{12} is the extended gradient of a conserved quantity of* (4.29), *γ_{11} is the gradient of a conserved quantity of* (4.29).

Proof. We first note that Eqs. (4.28) imply

a_1) $$\Phi_{12_f}[\ ]\delta_{12}g_{12}-\delta_{12}\Phi_{12_f}[\ ]g_{12}=0, \tag{4.30a}$$

a_2) $$\Phi_{12_d}[\delta_{12}\cdot]\delta_{12}g_{12}-\delta_{12}\Phi_{12_d}[\cdot]\delta_{12}g_{12}=0, \tag{4.30b}$$

a_3) $$(\delta_{12}\hat{K}^0_{12}\cdot 1)_f[\ ]=\delta_{12}(\hat{K}^0_{12}\cdot 1)_f[\ ], \tag{4.30c}$$

a_4) $$(\delta_{12}\hat{K}^0_{12}\cdot 1)_d[\delta_{12}\cdot]=\delta_{12}(\delta_{12}\hat{K}^0_{12}\cdot 1)_d[\cdot]. \tag{4.30d}$$

Equations (4.30a), (4.30b) follow from (4.28a) (see Appendix A). Using (4.28b) and the fact that $\hat{S}_{12_f}[\cdot]H_{12}=\hat{S}_{12_d}[\cdot]H_{12}=0$, Eqs. (4.30c), (4.30d) take the form of (4.30a), (4.30b) (with Φ_{12} replaced by $\hat{K}^0_{12}$). However, these equations follow from (4.28b) following a proof similar to the one given in the Appendix A.

a) Equations (4.28a), (4.30a), (4.30c) imply

$$(\delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1)_f[\ ]=\delta_{12}(\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1)_f[\ ]. \tag{4.31$_n$}$$

We derive Eq. $(4.31)_n$ by induction: Eq. $(4.31)_0$ is (4.30c). Let subscript L denote any derivative, such that the Leibnitz rule holds. Then

$$(\delta_{12}K_{12}^{(n+1)})_L=(\delta_{12}\Phi_{12}K_{12}^{(n)})_L=(\Phi_{12}\delta_{12}K_{12}^{(n)})_L+\beta(\delta'_{12}K_{12}^{(n)})_L.$$

Hence

$$(\delta_{12}K_{12}^{(n+1)})_L[\ ]=\Phi_{12_L}[\ ]\delta_{12}K_{12}^{(n)}+\Phi_{12}(\delta_{12}K_{12}^{(n)})_L[\ ]+\beta(\delta'_{12}K_{12}^{(n)})_L[\ ]. \tag{4.32}$$

We assume that $(4.31)_n$ is valid, then applying $\mathscr{D}$ on it, it follows that

$$(\delta'_{12}K_{12}^{(n)})_f[\ ]=\delta'_{12}K_{12_f}^{(n)}[\ ] \tag{4.33}$$

is also valid: To derive Eq. (4.33) note that Eqs. (4.26) imply

$$\mathscr{D}\delta_{12}\hat{\alpha}_{12}\cdot 1=\delta'_{12}\hat{\alpha}_{12}\cdot 1.$$

Applying the L-derivative on the above we obtain

$$\mathscr{D}(\delta_{12}\hat{\alpha}_{12}\cdot 1)_L[\ ]=(\delta'_{12}\hat{\alpha}_{12}\cdot 1)_L[\ ].$$

The above equation for $L=f$, and (4.26) imply (4.33). Equation $(4.31)_{n+1}$ is valid iff:

$$\begin{aligned}&\Phi_{12_f}[\ ]\delta_{12}G^n+\Phi_{12}(\delta_{12}G^n)_f[\ ]+\beta(\delta'_{12}G^n)_f[\ ]\\&=\delta_{12}\Phi_{12_f}[\ ]G^n+(\Phi_{12}\delta_{12}+\beta\delta'_{12})G_{n_f}[\ ].\end{aligned}$$

The first terms of the left- and right-hand sides of the above equation are equal because of (4.30a); the second and the third terms are equal because of $(4.31)_n$ and (4.33), respectively.

b) Equations (4.28a), (4.30b), (4.30d) imply

$$(\delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1)_d[\delta_{12}\cdot\ ]=\delta_{12}(\delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1)_d[\cdot\ ]. \tag{4.34$_n$}$$

To derive Eq. $(4.34)_n$ we use again induction. Equation $(4.34)_0$ is (4.30c). Assume that $(4.34)_n$ is valid, then applying the operator $\mathscr{D}$ on it, it follows that

$$(\delta'_{12}K_{12}^{(n)})_d[\delta_{12}\cdot\ ]=\delta_{12}(\delta'_{12}K_{12}^{(n)})_d[\cdot\ ]+\delta'_{12}(\delta_{12}K_{12}^{(n)})_d[\cdot\ ]. \tag{4.35}$$

Using (4.35) it follows that Eq. $(4.34)_{n+1}$ is valid if

$$\begin{aligned}&\Phi_{12_d}[\delta_{12}\cdot\ ]\delta_{12}K_{12}^{(n)}+\Phi_{12}(\delta_{12}K_{12}^{(n)})_d[\delta_{12}\cdot\ ]+\beta(\delta'_{12}K_{12}^{(n)})_d[\delta_{12}\cdot\ ]\\&\quad=\delta_{12}\Phi_{12_d}[\cdot\ ]\delta_{12}K_{12}^{(n)}+(\Phi_{12}\delta_{12}+\beta\delta'_{12})(\delta_{12}K_{12}^{(n)})_d[\cdot\ ]+\delta_{12}\beta(\delta'_{12}K_{12}^{(n)})_d[\cdot\ ].\end{aligned}$$

The first term of the left- and right-hand sides of the above equation are valid because of (4.30b); the second and the remainder terms because of $(4.34)_n$ and (4.35), respectively.

c) Equations (4.28), (4.30), $(4.34)_n$, $(4.31)_n$, and (4.6) imply:

$$\begin{aligned}\delta_{12}(\delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1)_d[\cdot\ ]&=(\delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1)_d[\delta_{12}\cdot\ ]=(\delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1)_f[\cdot\ ]\\&=\delta_{12}(\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1)_f[\cdot\ ].\end{aligned} \tag{4.36}$$

Using the definitions of symmetries and extended symmetries and Eq. (4.30c–d), the first part of Theorem 4.1 follows:

$$\sigma_{11_t} = \int_{\mathbb{R}} dy_2 \delta_{12} \sigma_{12_t} = \int_{\mathbb{R}} dy_2 \delta_{12} (\delta_{12} \Phi_{12}^n \mathcal{R}_{12}^0 \cdot 1)_d [\sigma_{12}]$$

$$= \int_{\mathbb{R}} dy_2 \delta_{12} (\Phi_{12}^n \mathcal{R}_{12}^0 \cdot 1)_f [\sigma] = K_{11_f}^{(n)} [\sigma_{11}] .$$

The derivation of ii) is similar to the derivation of i): It follows from the equations

$$(\delta_{12} \Phi_{12}^n \mathcal{R}_{12}^0 \cdot 1)_f^* [\ ] = \delta_{12} (\Phi_{12}^n \mathcal{R}_{12}^0 \cdot 1)_f^* [\ ] , \tag{4.37a}$$

$$(\delta_{12} \Phi_{12}^n \mathcal{R}_{12}^{(0)} \cdot 1)_d^* [\delta_{12} \cdot] = \delta_{12} (\delta_{12} \Phi_{12}^n \mathcal{R}_{12}^0 \cdot 1)_d^* [\cdot] , \tag{4.37b}$$

which are direct consequences of Eqs. $(4.31)_n$, $(4.34)_n$, (4.6), (4.7), and (4.8). Then

$$\begin{aligned} \gamma_{11_t} &= \int_{\mathbb{R}} dy_2 \delta_{12} \gamma_{12_t} = - \int_{\mathbb{R}} dy_2 \delta_{12} (\delta_{12} \Phi_{12}^n \mathcal{R}_{12}^0 \cdot 1)_d^* [\gamma_{12}] \\ &= - \int_{\mathbb{R}} dy_2 (\delta_{12} \Phi_{12}^n \mathcal{R}_{12}^0 \cdot 1)_d^* [\delta_{12} \gamma_{12}] = - \int_{\mathbb{R}} dy_2 (\delta_{12} \Phi_{12}^n \mathcal{R}_{12}^0 \cdot 1)_f^* [\gamma] \\ &= - \int_{\mathbb{R}} dy_2 \delta_{12} (\Phi_{12}^n \mathcal{R}_{12}^0 \cdot 1)_f^* [\gamma] = - K_{11_f}^{(n)*} [\gamma] . \end{aligned}$$

The derivation of iii) follows from ii) and the fact that if γ_{12} is an extended gradient function γ_{11} is a gradient function: Recall that γ_{12} is an extended gradient iff $\gamma_{12_d}[\ ] = \gamma_{12_d}^*[\ ]$, namely iff $\langle \gamma_{12_d}[g_{12}], f_{12} \rangle = \langle g_{12}, \gamma_{12_d}[f_{12}] \rangle$. Letting $f_{12} \to \delta_{12} f_{12}$ and $g_{12} \to \delta_{12} g_{12}$, we obtain $(\gamma_{11_f}[g_{11}], f_{11}) = (g_{11}, \gamma_{11_f}[f_{11}])$ which implies that $\gamma_{11_f} = \gamma_{11_f}^+$ (γ_{11} is a gradient). Moreover, one could easily show that if $\gamma_{12} = \mathrm{grad}_{12} I$, then $\gamma_{11} = \mathrm{grad}\, I$.

Another important property of the extended symmetries is given by the following theorem:

Theorem 4.2. *If σ_{12} is an extended symmetry of Eq. (4.29), then $\sigma_{12} = 0$ is an auto-Bäcklund Transformation for Eq. (4.29). In equation $\sigma_{12} = 0$, q_1 and q_2 are viewed as two different solutions of (4.29).*

Proof. If σ_{12} is an extended symmetry of Eq. (4.29) and $\sigma_{12} = 0$, then $D_t \sigma_{12} = \dfrac{\partial \sigma_{12}}{\partial t} + \sigma_{12_f}[K] = 0$, which implies the result.

Remark 4.5. Theorems 4.1 and 4.2 show that the symmetries and the auto-Bäcklund Transformations of an equation originate from the same entity: the extended symmetry. This remarkable connection between symmetries and auto-Bäcklund Transformations exists also in $1+1$ dimensions. If we consider as an example the classes of evolution equations in $2+1$ dimensions (3.19), (3.17), (3.35), and (3.38), then extended symmetries and gradients for the corresponding $1+1$ dimensional systems are still defined by Eqs. (4.17) and (4.18), in which the operators $(\delta_{12} K_{12})_d$ and $(\delta_{12} K_{12})_d^*$ are evaluated at $\alpha = 0$. For $\alpha = 0$ Φ_{12} is indeed the operator that generates Bäcklund Transformations in $1+1$ dimensions [38].

The above theorems imply that it is useful to have an effective way of generating extended symmetries and extended gradients of conserved quantities.

For equations in $1+1$ one makes fundamental use of the following two notions: a) if Φ is hereditary it generates infinitely many commuting symmetries. b) If Φ admits a factorization in terms of compatible Hamiltonian operators it generates infinitely many constants of motion in involution. Both the above notions are extended to equations in $2+1$.

B. Characterization of the Starting Symmetry $\hat{K}^0_{12}\cdot H_{12}$ through the Recursion Operator Φ_{12}

Fundamental role in the theory presented in this paper is played by a hereditary operator Φ_{12} and a starting symmetry $\hat{K}^0_{12}H_{12}$. It is interesting that the recursion operator Φ_{12} algorithmically implies $\hat{K}^0_{12}H_{12}$. Furthermore, if Φ_{12} is hereditary, it is also a strong symmetry for $\hat{K}^0_{12}H_{12}$.

Definition 4.3. A starting symmetry associated with the recursion operator Φ_{12} is $\hat{K}^0_{12}H_{12}$, where the admissible operator $\hat{K}^0_{12}$ and the function H_{12} satisfy

$$\Phi_{12}\hat{S}_{12}\cdot H_{12}=\hat{K}^0_{12}H_{12}, \qquad \hat{S}_{12}\cdot H_{12}=0, \tag{4.38}$$

and $\hat{S}_{12}$ is an invertible operator, of course, on a space of functions excluding $\mathrm{Ker}\,\hat{S}_{12}\ni H_{12}$.

Examples. 1. For the KP hierarchies, $\hat{S}_{12}=D$ and/or $\hat{S}_{12}=D(q^-_{12})^{-1}D$. This implies

$$\hat{K}^0_{12}=Dq^+_{12}+q^-_{12}D^{-1}q^-_{12}, \qquad \hat{S}_{12}=D, \tag{4.39a}$$

$$\hat{K}^0_{12}=q^-_{12}, \qquad \hat{S}_{12}=D(q^-_{12})^{-1}D, \tag{4.39b}$$

with H_{12} any solution of $DH_{12}=0$.

2. For the DS hierarchies $\hat{S}_{12}=(Q^+_{12})^{-1}P_{12}$. This implies

$$\hat{K}^0_{12}=Q^-_{12}\sigma \quad \text{and/or} \quad \hat{K}^0_{12}=Q^-_{12}, \tag{4.40}$$

with H_{12} any diagonal matrix solving $P_{12}H_{12}=0$.

For the results presented in this paper we only use a subclass of solutions of $DH_{12}=0$ and $P_{12}H_{12}=0$, given by $H_{12}=h_{12}\doteq h(y_1-y_2)$ and $H_{12}=h_{12}(aI+b\sigma)$, a, b constants, respectively. More general solutions of the above equations are used in [35] and give rise to time-dependent symmetries.

Lemma 4.2. *If $\hat{K}^0_{12}H_{12}$ is a starting symmetry associated with the hereditary operator Φ_{12}, then Φ_{12} is a strong symmetry of $\hat{K}^0_{12}H_{12}$.*

Proof. Since Φ_{12} is hereditary,

$$\Phi_{12_d}[\Phi_{12}f_{12}]g_{12}-\Phi_{12}\Phi_{12_d}[f_{12}]g_{12} \quad \text{is symmetric in } f_{12}, g_{12}. \tag{4.41}$$

Letting $g_{12}=\hat{S}_{12}\cdot H_{12}$ we obtain

$$\Phi_{12_d}[\Phi_{12}\hat{S}_{12}H_{12}]f_{12}-\Phi_{12}\Phi_{12_d}[\hat{S}_{12}H_{12}]f_{12}-\Phi_{12_d}[\Phi_{12}f_{12}]\hat{S}_{12}H_{12}$$
$$+\Phi_{12}\Phi_{12_d}[f_{12}]\hat{S}_{12}H_{12}=0.$$

Using $\Phi_{12}\hat{S}_{12}H_{12}=\hat{K}^0_{12}H_{12}$, $\hat{S}_{12}H_{12}=0$ and its consequence $\hat{S}_{12_d}[f_{12}]H_{12}=0$, for every f_{12}, we obtain

$$\Phi_{12_d}[\hat{K}^0_{12}H_{12}]f_{12}-(\hat{K}^0_{12}H_{12})_d[\Phi_{12}f_{12}]+\Phi_{12}(\hat{K}^0_{12}H_{12})_d[f_{12}]=0, \quad \forall f_{12}, \tag{4.42}$$

thus Φ_{12} is a strong symmetry of $\hat{K}^0_{12}H_{12}$.

C. Hereditary Symmetries

Theorem 4.3. *Assume that the admissible hereditary operator Φ_{12} and its associated starting symmetry $\hat{K}^0_{12}H_{12}$, defined via*

$$\Phi_{12}\hat{S}_{12}H_{12}=\hat{K}^0_{12}H_{12}, \quad \hat{S}_{12}H_{12}=0 \tag{4.43}$$

satisfy

$$[\Phi_{12},h_{12}]=-\beta h'_{12}, \tag{4.44a}$$

$$[\hat{K}^0_{12},h_{12}]=-\bar{\beta}\hat{S}_{12}h'_{12}, \tag{4.44b}$$

where $\beta,\bar{\beta}$ are constants, $\hat{S}_{12}$ is an admissible operator, $h_{12}=h(y_1-y_2)$ and prime denotes derivative with respect to y_1. Further assume that

$$[\hat{K}^0_{12}H^{(1)}_{12},\hat{K}^0_{12}H^{(2)}_{12}]_d=0, \quad \textit{for} \quad [H^{(1)}_{12},H^{(2)}_{12}]_I=0, \tag{4.44c}$$

where $[\ ]_d$, $[\ ]_I$ are defined by (4.3) and h_{12} belongs to H_{12}. Then

$$[\Phi^m_{12}\hat{K}^0_{12}H^{(1)}_{12},\Phi^n_{12}\hat{K}^0_{12}H^{(2)}_{12}]_d=0, \quad \textit{for} \quad [H^{(1)}_{12},H^{(2)}_{12}]_I=0. \tag{4.45a}$$

Furthermore,

$$\Phi^m_{12}\hat{K}^0_{12}\cdot 1 \quad \textit{are extended symmetries of } (4.4)_n, \tag{4.45b}$$

for all nonnegative integers m, n.

Proof. In analogy with the results of 1+1 one easily verifies that if $K^{(1)}_{12}$, $K^{(2)}_{12}$ commute, Φ_{12} is hereditary and Φ_{12} is a strong symmetry for both $K^{(1)}_{12}$ and $K^{(2)}_{12}$, then $\Phi^m_{12}K^{(1)}_{12}$, $\Phi^m_{12}K^{(2)}_{12}$ also commute, for all m, n. Using these results with $K^{(1)}_{12}=\hat{K}^0_{12}H^{(1)}_{12}$, $K^{(2)}_{12}=\hat{K}^0_{12}H^{(2)}_{12}$ one immediately proves (4.45a) above. To prove (4.45b) we note that (4.44) imply

$$\delta_{12}K^{(n)}_{12}=\sum_{\ell=0}^{n} b_{n,\ell}\Phi^{n-\ell}_{12}\hat{K}^0_{12}\delta^\ell_{12}, \tag{4.46}$$

where $b_{n,\ell}$ depend on $\beta,\bar{\beta}$ (see Appendix B). Hence

$$[\Phi^m_{12}\hat{K}^0_{12}\cdot 1,(\Phi_{12}+\beta\mathscr{D})^n\delta_{12}\hat{K}^0_{12}\cdot 1]_d=\left[\Phi^m_{12}\hat{K}^0_{12}\cdot 1,\ \sum_{\ell=0}^{n} b_{n,\ell}\Phi^{n-\ell}_{12}\hat{K}^0_{12}\cdot\delta^\ell_{12}\right]_d=0. \tag{4.47}$$

Equation (4.47) follows from (4.45a) since $[1,\delta^\ell_{12}]_I=0$ for all nonnegative integers ℓ. The left-hand side of Eq. (4.47) equals

$$(\Phi^m_{12}\hat{K}^0_{12}\cdot 1)_d[\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1]-(\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_d[\Phi^m_{12}\hat{K}^0_{12}\cdot 1];$$

but the first term of the above equals $(\Phi^m_{12}\hat{K}^0_{12}\cdot 1)_f[K^{(n)}]$, hence (4.45b) follows.

It turns out that the recursion operators associated with both the two-dimensional Schrödinger and the two-dimensional 2×2 AKNS are hereditary. Actually, isospectral eigenvalue equations always yield hereditary operators (see Sect. 4E).

Remark 4.6. If Φ_{12} generates two classes of evolution equations $(4.4)_n$, corresponding to two different starting points $\hat{M}_{12}$ and $\hat{N}_{12}$, and if, in addition to (4.44), we have

$$[\hat{M}_{12}H_{12}^{(1)}, \hat{N}_{12}H_{12}^{(2)}]_d = 0, \quad \text{for} \quad [H_{12}^{(1)}, H_{12}^{(2)}]_l = 0, \tag{4.48}$$

then $\Phi_{12}^m \hat{M}_{12} \cdot 1$ and $\Phi_{12}^m \hat{N}_{12} \cdot 1$ are extended symmetries for both classes of evolution equations.

D. Bi-Hamiltonian Systems

Definition 4.4. i) An admissible operator Θ_{12} is called a Hamiltonian (inverse symplectic) operator iff

a) $$\Theta_{12}^* = -\Theta_{12}, \tag{4.49a}$$

b) it satisfies the Jacobi identity with respect to the bracket

$$\{a_{12}, b_{12}, c_{12}\} \doteq \langle a_{12}, \Theta_{12_d}[\Theta_{12} b_{12}] c_{12} \rangle, \tag{4.49b}$$

for arbitrary a_{12}, b_{12}, c_{12}.

ii) An Eq. (4.16) is of a Hamiltonian form (or is a Hamiltonian system) if it can be written as

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} \Theta_{12} \gamma_{12}, \tag{4.50}$$

where Θ_{12} is a Hamiltonian operator and γ_{12} is an extended gradient function of the form $\gamma_{12} = \hat{\gamma}_{12} \cdot 1$ [with, of course, $(\hat{\gamma}_{12} H_{12})_d = (\hat{\gamma}_{12} H_{12})_d^*$].

The associated Poisson bracket is given by:

$$\{I^{(1)}, I^{(2)}\}_H \doteq \langle \mathrm{grad}_{12} I^{(1)}, \Theta_{12}\, \mathrm{grad}_{12} I^{(2)} \rangle, \tag{4.51}$$

where the functional $I^{(i)}$ is given by $I^{(i)} = \int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} \hat{\varrho}_{12}^{(i)} H_{12}^{(i)}$.

Remark 4.7. If Θ_{12} satisfies a), b) above then the Poisson bracket (4.51) is skew symmetric and satisfies the Jacobi identity.

Proposition 4.1. *Let*

$$G_{12} = \Theta_{12} f_{12}, \quad \Theta_{12} \textit{ skew symmetric}. \tag{4.52}$$

Then for arbitrary a_{12}, b_{12} the following identities are valid.

$$\begin{aligned} \mathrm{a}_1)\ & \langle b_{12}, (\Theta_{12_d}[G_{12}] - \Theta_{12}(G_{12})_d^* - (G_{12})_d^* \Theta_{12}) a_{12} \rangle \\ &= \{b_{12}, f_{12}, a_{12}\} + \{f_{12}, a_{12}, b_{12}\} + \{a_{12}, b_{12}, f_{12}\} \\ &\quad + \langle b_{12}, \Theta_{12}(f_{12_d} - f_{12_d}^*) \Theta_{12} a_{12} \rangle . \end{aligned} \tag{4.53}$$

Let Θ_{12} be Hamiltonian and let a_{12}, b_{12} be extended gradient functions. Then

$$\text{a}_2)\qquad [\Theta_{12}a_{12}, \Theta_{12}b_{12}]_d = \Theta_{12}\,\text{grad}_{12}\langle a_{12}, \Theta_{12}b_{12}\rangle. \tag{4.54}$$

These identities imply:

a$_3$) *If Θ_{12} is a Hamiltonian operator and f_{12} is an extended gradient, then Θ_{12} is a Noether operator for G_{12}.*

a$_4$) *If Θ_{12} is a Hamiltonian operator and it is a Noether operator for G_{12} then f_{12} is an extended gradient function.*

The above results are exactly analogous to those in $1+1$ and thus their derivation is omitted.

The above results can be used for any Hamiltonian system as soon as the commutator $[\Theta_{12}, H_{12}]$ is specified. However, for a completely integrable Hamiltonian system additional results are valid.

Proposition 4.2. *Let*

$$\hat{\gamma}_{12}^{(m)} \doteq (\Phi_{12}^{*})^m \Theta_{12}^{-1} \hat{K}_{12}^0, \qquad \gamma_{12}^{(m)} \doteq \hat{\gamma}_{12}^{(m)}\cdot 1, \qquad \hat{K}_{12}^{(n)} \doteq \Phi_{12}^n \hat{K}_{12}^0. \tag{4.55}$$

Assume that Θ_{12} is Hamiltonian, its inverse exists and that $\hat{\gamma}_{12}^{(m)}H_{12}$ are extended gradients. Further assume that Eqs. (4.4) are valid. Then

$$\text{i)}\qquad \langle \hat{\gamma}_{12}^{(m)}H_{12}^{(1)}, \hat{K}_{12}^{(n)}H_{12}^{(2)}\rangle = \langle \hat{\gamma}_{12}^{(m)}H_{12}^{(1)}, \Theta_{12}\hat{\gamma}_{12}^{(n)}H_{12}^{(2)}\rangle = 0, \tag{4.56}$$

$$\text{ii)}\qquad (\gamma_{11}^{(m)}, K_{11}^{(n)}) = 0, \quad \text{if} \quad [H_{12}^{(1)}, H_{12}^{(2)}]_t = 0. \tag{4.57}$$

Proof. Since the hereditary operator Φ_{12} is a strong symmetry for the starting symmetry $\hat{K}_{12}^0 H_{12}$ that satisfies (4.4c), then $[\hat{K}_{12}^{(m)}H_{12}^{(1)}, \hat{K}_{12}^{(n)}H_{12}^{(2)}]_d = 0$ if $[H_{12}^{(1)}, H_{12}^{(2)}]_t = 0$. Then (4.56) follows from Proposition 4.1a$_2$). Equation (4.57) follows from (4.56) choosing $H_{12}^{(1)} = 1$ and $H_{12}^{(2)} = \delta_{12}^{\prime}$:

$$(\gamma_{11}^{(m)}, K_{11}^{(n)}) = \langle \gamma_{12}^{(m)}, \delta_{12}K_{12}^{(n)}\rangle = \Big\langle \hat{\gamma}_{12}^{(m)}\cdot 1, \sum_{s=0}^{n} b_{n,s}\Phi_{12}^{n-s}\hat{K}_{12}^0\delta_{12}^s\Big\rangle = 0.$$

Theorem 4.4. *Let $\Theta_{12}^{(1)}$, $\Theta_{12}^{(2)}$, $\Theta_{12}^{(1)} + \Theta_{12}^{(2)}$ be Hamiltonian operators and assume that $\Theta_{12}^{(1)}$ is invertible. Then*

i) *$\Phi_{12} = \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$ is a hereditary operator.*

ii) *$\Phi_{12}^n\Theta_{12}^{(1)}$, are Hamiltonian operators.*

iii) *If $\hat{\gamma}_{12}^0 H_{12} \doteq (\Theta_{12}^{(1)})^{-1}\hat{K}_{12}^0 H_{12}$ is an extended gradient function and if Eqs. (4.44) hold, then Eq. (4.4)$_n$ is a bi-Hamiltonian system having $\Theta_{12}^{(1)}, \Theta_{12}^{(2)}$ as Noether operators.*

Furthermore, all functions $\gamma_{12}^{(m)}$

$$\gamma_{12}^{(m)} \doteq \hat{\gamma}_{12}^{(m)}\cdot 1, \qquad \hat{\gamma}_{12}^{(m)} \doteq (\Theta_{12}^{(1)})^{-1}\hat{K}_{12}^{(m)}, \qquad \hat{K}_{12}^{(m)} \doteq \Phi_{12}^m \hat{K}_{12}^0 \tag{4.58}$$

are extended gradients of conserved quantities in involution under the two Poisson brackets defined by

$$\{I^{(m)}, I^{(n)}\} \doteq \langle \delta_{12}\gamma_{12}^{(m)}, \Theta_{12}\gamma_{12}^{(n)}\rangle, \qquad \Theta_{12} = \Theta_{12}^{(1)} \text{ or } \Theta_{12}^{(2)}. \tag{4.59}$$

Proof. The derivation of the above results is analogous to similar results for equations in $1+1$ (see for example [7]). With respect to iii) above we note that $\hat{K}_{12}^{(n)}H_{12} = \Phi_{12}^n\Theta_{12}^{(1)}\hat{\gamma}_{12}^0 H_{12}$, hence $\Phi_{12}^n\Theta_{12}^{(1)}$ is a Noether operator for $\Phi_{12}^n\hat{K}_{12}^0 H_{12}$;

the arbitrariness of H_{12} and (4.46) imply that $\Phi_{12}^n\Theta_{12}^{(1)}$ is a Noether operator for $(4.4)_n$; hence $(4.4)_n$ is a Hamiltonian system with $\Phi_{12}^n\Theta_{12}^{(1)}$ as a Noether operator. However, Φ_{12} is a strong symmetry for $\hat{K}_{12}^0 H_{12}$, hence Φ_{12}^n is a strong symmetry for $\hat{K}_{12}^0 H_{12}$. Since $\Phi_{12}^n\Theta_{12}^{(1)}$ is Noether and Φ_{12}^n is a strong symmetry $\Theta_{12}^{(1)}$ is also Noether. Thus $\Theta_{12}^{(2)}=\Phi_{12}\Theta_{12}^{(1)}$ is also a Noether operator. Furthermore, $K_{12}^{(n)}=\Phi_{12}^{n-m}\Theta_{12}^{(1)}\gamma_{12}^{(m)}$, and the operator $\Phi_{12}^{n-m}\Theta_{12}^{(1)}$ is both Noether and Hamiltonian, thus $\gamma_{12}^{(m)}H_{12}$ are extended gradient functions (using Proposition 4.1).

It now trivially follows [since Theorem 4.3 implies that $K_{12}^{(m)}$ are extended symmetries of $(4.4)_n$] that $\gamma_{12}^{(m)}$ are conserved covariants of $(4.4)_n$. Moreover, Proposition 4.2 implies:

$$\begin{aligned}\{I^{(m)},I^{(n)}\}_H&=\langle\gamma_{12}^{(m)}H_{12}^{(1)},\Theta_{12}^{(1)}\gamma_{12}^{(n)}H_{12}^{(2)}\rangle\\&=\langle\gamma_{12}^{(m)}H_{12}^{(1)},\Theta_{12}^{(2)}\gamma_{12}^{(n-1)}H_{12}^{(2)}\rangle=0\,,\quad\text{if}\quad[H_{12}^{(1)},H_{12}^{(2)}]_t=0\,,\end{aligned}$$

and the choice $H_{12}^{(1)}=\delta_{12}^{(\ell)}$, $H_{12}^{(2)}=1$ yields

$$\{I^{(m)},I^{(n)}\}\doteq\langle\delta_{12}\gamma_{12}^{(m)},\Theta_{12}\gamma_{12}^{(n)}\rangle=0\,,\qquad\Theta_{12}=\Theta_{12}^{(1)}\text{ or }\Theta_{12}^{(2)}\,.\tag{4.60a}$$

Namely $\gamma_{12}^{(n)}$, are extended gradients of conserved quantities in involution. If $[\Theta_{12},\delta_{12}]=0$, then

$$(\gamma_{11}^{(m)},\Theta_{11}\gamma_{11}^{(n)})=0\,.\tag{4.60b}$$

Combining Theorems 4.1–4.4, we obtain the following important theorem.

Theorem 4.5. *Let $\Theta_{12}^{(1)}+v\Theta_{12}^{(2)}$ be a Hamiltonian operator for all constant values of v. Assume that $\Theta_{12}^{(1)}$ is invertible. Define*

$$\Phi_{12}\doteq\Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}\,,\qquad K_{12}^{(n)}\doteq\Phi_{12}^n\hat{K}_{12}^0\cdot 1\,,\qquad\gamma_{12}^0\doteq(\Theta_{12}^{(1)})^{-1}K_{12}^0\,.\tag{4.61}$$

Assume that the operator Φ_{12} and its associated starting symmetry $\hat{K}_{12}^0 H_{12}$ satisfy (4.44). Further assume that $\gamma_{12}^{(0)}$ is an extended gradient function. Then

i) *Equations $(4.4)_n$ are bi-Hamiltonian systems.*

ii) *$K_{12}^{(m)}\doteq\Phi_{12}^m\hat{K}_{12}^0\cdot 1$, $\gamma_{12}^{(m)}=(\Phi_{12}^*)^m\gamma_{12}^0$ are extended symmetries and extended gradients of conserved quantities, respectively, for Eq. $(4.4)_n$.*

iii) *$K_{11}^{(m)}$ and $\gamma_{11}^{(m)}$ are symmetries and gradients of conserved quantities in involution for $q_{1_t}=K_{11}^{(n)}$.*

iv) *$K_{12}^{(m)}=0$ are auto-Bäcklund Transformations for Eq. $(4.4)_n$.*

v)
$$[K_{11}^{(m)},K_{11}^{(n)}]_f=0\,,\tag{4.62a}$$

$$\{I^{(m)},I^{(n)}\}\doteq\langle\delta_{12}\gamma_{12}^{(m)},\Theta_{12}\gamma_{12}^{(n)}\rangle=0\,,\qquad\Theta_{12}=\Theta_{12}^{(1)}\text{ or }\Theta_{12}^{(2)}\,,\tag{4.62b}$$

where

$$[a,b]_f=a_f[b]-b_f[a]\,.\tag{4.62c}$$

E. Isospectral Problems Yield Hereditary Operators

Section 4.C illustrates the importance of hereditary operators. For equations in 1+1, isospectral problems yield hereditary operators. A similar construction is possible for equations in 2+1. Furthermore, this construction also provides us with a simple commutation relation of the type (4.24a) between Φ_{12} and h_{12}.

Proposition 4.3. *Let*

$$\frac{dV}{dx} = U(\hat{q}, \lambda)V \tag{4.63}$$

be an isospectral two-dimensional problem; $\hat{q}$ is an operator depending on $q(x, y)$ and $\partial/\partial y$; λ is an eigenvalue. Assume that $(G_\lambda)_{12}$, the extended gradient of λ satisfies

$$\Psi_{12}(G_\lambda)_{12} = \mu(\lambda)(G_\lambda)_{12}\,. \tag{4.64}$$

Then if $\Phi_{12} \doteqdot \Psi^*_{12}$ has a complete set of eigenfunctions, it is hereditary operator.

Instead of deriving this result we illustrate it by two examples. The interested reader is referred to [5]. A proof of completeness should follow a two-dimensional version of the method developed by [10].

The derivation of Eq. (4.24a) from Eqs. (4.63) and (4.64) is also illustrated in an example.

Example 1. Consider the isospectral problem

$$v_{1xx} + (q_1 + \alpha D_{y_1})v_1 = \lambda v_1\,. \tag{4.65}$$

Let $\hat{q}_1 \doteqdot q_1 + \alpha D_{y_1}$ and consider the directional derivative of (4.65):

$$v_{1_{xx_d}}[\;] + \hat{q}_{1_d}[\;]v_1 + \hat{q}_1 v_{1_d}[\;] = \lambda v_{1_d}[\;] + \lambda_d[\;]v_1\,.$$

Multiplying the above by v_1^+, where v_1^+ satisfies the adjoint of (4.65), with respect to the bilinear form (4.9), integrating with respect to $dy_1 dx$, and assuming $\int_{\mathbb{R}^2} dx dy_1 v_1 v_1^+ = 1$ it follows that

$$\lambda_d[f_{12}] = \int_{\mathbb{R}^2} dx dy_1 v_1^+ \hat{q}_{1_d}[f_{12}]v_1\,. \tag{4.66}$$

Using (4.1b) to evaluate $\hat{q}_{1_d}[f_{12}]v_1$ it follows that

$$\lambda_d[f_{12}] = \int_{\mathbb{R}^3} dx dy_1 dy_2 v_2 v_1^+ f_{12}\,.$$

Hence, using $\lambda_d[f_{12}] = \int_{\mathbb{R}^3} dx dy_1 dy_2 (\operatorname{grad}\lambda)_{21} f_{12}$, it follows that

$$(\operatorname{grad}\lambda)_{12} = v_1 v_2^+\,. \tag{4.67}$$

Since Φ_{12} defined by (1.2a) satisfies [29]

$$\Phi^*_{12} v_1 v_2^+ = 4\lambda v_1 v_2^+\,, \tag{4.68}$$

it follows that Φ_{12} is hereditary.

Example 2. Consider the isospectral problem

$$V_{1_x} - JV_{1_y} - Q_1 V_1 = \lambda J V_1\,, \tag{4.69}$$

where J, Q are defined in (1.8). In analogy with (4.66) and assuming $\operatorname{tr}\int_{\mathbb{R}^2} dx dy_1 V_1^+ J V_1 = 1$, we find

$$\lambda_d[F_{12}] = \operatorname{tr}\int_{\mathbb{R}^2} dx dy_1 V_1^+ \hat{Q}_{1_d}[F_{12}]V_1\,.$$

Hence, using $\hat{Q}_{1_d}[F_{12}]G_{12} = \int_{\mathbf{R}} dy_3 F_{13} G_{32}$, it follows that

$$\lambda_d[F_{12}] = \operatorname{tr} \int_{\mathbf{R}^3} dx dy_1 dy_2 V_1^+ F_{12} V_2 .$$

Thus

$$(\operatorname{grad} \lambda)_{12} = V_1 V_2^+ .$$

Since $R_{12} \doteq D - \hat{Q}_{12}$ satisfies

$$R_{12} V_1 V_2^+ = \lambda \hat{J} V_1 V_2^+ , \quad \hat{J} F_{12} \doteq J F_{12} - F_{12} J , \tag{4.70}$$

it follows that $(R_{12}^{-1} \hat{J})^* = \hat{J}^* (R_{12}^{-1})^* = \hat{J} R_{12}^{-1}$ is hereditary (see [39] for the analogous result in $1+1$ dimensions).

Now we show that Eqs. (4.65) and (4.68) imply

$$[\Phi_{12}, h_{12}] = 4\alpha h'_{12} , \quad h_{12} = h(y_1 - y_2) . \tag{4.71}$$

First, we recall that Eq. (4.68) follows from Eq. (4.65): Eq. (4.68) and its adjoint $V^+_{2_{xx}} + (q_2 - \alpha D_2) V_2^+ = \lambda V_2^+$ imply

$$V_{1_{xx}} V_2^+ + (q_1 + \alpha D_1) V_1 V_2^+ = \lambda V_1 V_2^+ , \tag{4.72a}$$

$$V_1 V^+_{2_{xx}} + (q_2 - \alpha D_2) V_1 V_2^+ = \lambda V_1 V_2^+ , \tag{4.72b}$$

$$V_{1_{xx}} V^+_{2_x} + (q_1 + \alpha D_1) V_1 V^+_{2_x} = \lambda V_1 V^+_{2_x} , \tag{4.73a}$$

$$V_{1_x} V^+_{2_{xx}} + (q_2 - \alpha D_2) V_{1_x} V_2^+ = \lambda V_{1_x} V_2^+ . \tag{4.73b}$$

Adding Eqs. (4.72a) and (4.72b), Eqs. (4.73a) and (4.73b), and subtracting Eq. (4.72b) from Eq. (4.72a) we obtain, respectively,

$$(D^2 + q^+_{12}) V_1 V_2^+ = 2 V_{1_x} V^+_{2_x} + 2\lambda V_1 V_2^+ , \tag{4.74a}$$

$$V_{1_x} V^+_{2_x} = -\frac{D^{-1}}{2} q^+_{12} D V_1 V_2^+ - \frac{D^{-1}}{2} q^-_{12} (V_1 V^+_{2_x} - V_{1_x} V_2^+) + \lambda V_1 V_2^+ , \tag{4.74b}$$

$$V_1 V^+_{2_x} - V_{1_x} V_2^+ = D^{-1} q^-_{12} V_1 V_2^+ . \tag{4.74c}$$

Using Eqs. (4.74b–c) into Eq. (4.74a) we finally obtain the eigenvalue equation (4.68).

Now, by virtue of the commutation relations $[q_1 + \alpha D_1, h_{12}] = [q_2 - \alpha D_2, h_{12}] = \alpha h'_{12}$, Eqs. (4.72) and (4.73) are still valid replacing $V_1 \to V_{12} \doteq h_{12} V_1$, $V_2^+ \to V^+_{12} \doteq h_{12} V_2^+$ and $\lambda \to \lambda_{12} \doteq \lambda + 2\alpha h'_{12}/h_{12}$; then $\Phi^*_{12} V_{12} V^+_{12} = 4\lambda_{12} V_{12} V^+_{12}$, namely

$$\begin{aligned} \Phi^*_{12} V_{12} V^+_{12} &= \Phi^*_{12} h^2_{12} V_1 V_2^+ = (h^2_{12} \Phi^*_{12} + [\Phi^*_{12}, h^2_{12}]) V_1 V_2^+ \\ &= (4\lambda h^2_{12} + 8\alpha h'_{12}/h_{12}) V_1 V_2^+ . \end{aligned}$$

Using Eq. (4.68) and the completeness of the eigenfunctions of Φ^*_{12}, Eq. (4.71) follows.

5. Applications

In this section we apply the theory developed in the previous sections to the classes of evolutions associated with the Schrödinger eigenvalue problem (1.1) and with the 2×2 AKNS problem (1.8).

Some interesting details of the explicit calculations concerning these two examples are separately presented in Appendix C.

An isospectral problem [e.g. (1.1)] yields a recursion operator Φ_{12} [e.g. (1.2a)]. This operator must be hereditary (see Sect. 4.E). The isospectral problem also yields a basic operator $\hat{q}_{12}$; the integral representation of this operator implies a directional derivative $\hat{q}_{1_d}$. Using the bilinear form (4.7), $\hat{q}_1^*$, $\hat{q}_{1_d}^*$ are also obtained.

i) In investigating the time-independent symmetries of the hierarchies associated with Φ_{12} one then needs to: a) Find the starting symmetries $\hat{K}_{12}^0 H_{12}$ associated with Φ_{12} (see Sect. 4.B). b) Calculate the commutator relations of Φ_{12}, $\hat{K}_{12}^0$ with h_{12}. c) Compute the Lie algebra of the starting symmetries. Then Theorems 4.1, 4.3 yield hierarchies of infinitely many commuting symmetries.

ii) In investigating the Hamiltonian nature of the hierarchies associated with Φ_{12} one, in addition to the above, also needs to: a) Prove that $\Theta_{12}^{(1)}, \Theta_{12}^{(2)}$, where $\Phi_{12} = \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$, are compatible Hamiltonian operators. b) Verify that the starting covariants are extended gradients. Then Theorem 4.4 yields hierarchies of infinitely many involutionary conserved quantities.

A. The Schrödinger Eigenvalue Problem

The spectral problem (1.1) yields the hereditary operator

$$\Phi_{12} = D^2 + q_{12}^+ + Dq_{12}^+ D^{-1} + q_{12}^- D^{-1} q_{12}^- D^{-1}, \tag{5.1a}$$

where

$$q_{12}^{\pm} \doteq q_1 \pm q_2 + \alpha(D_1 \mp D_2). \tag{5.1b}$$

The integral representation of the basic operator $\hat{q}_1$ implies an appropriate directional derivative:

$$\hat{q}_1 f_{12} \doteq (q_1 + \alpha D_1) f_{12} = \int_{\mathbb{R}} dy_3 q_{13} f_{32}, \quad \hat{q}_{1_d}[\sigma_{12}] f_{12} = \int_{\mathbb{R}} dy_3 \sigma_{13} f_{32}. \tag{5.2}$$

The adjoint of Eq. (5.2) implies

$$\hat{q}_1^* f_{12} = (q_1 - \alpha D_2) f_{12} = \int_{\mathbb{R}} dy_3 f_{13} q_{32}, \quad \hat{q}_{1_d}^*[\sigma_{12}] f_{12} = \int_{\mathbb{R}} dy_3 f_{13} \sigma_{32}. \tag{5.3}$$

Combining the above we obtain the following derivative:

$$\begin{aligned} a_{12}(\hat{q})_d[f_{12}] &= \frac{\partial}{\partial \varepsilon} a_{12}(q_{12}^{\pm} + \varepsilon f_{12}^{\pm})\Big|_{\varepsilon=0}, \\ f_{12}^{\pm} g_{12} &= \int_{\mathbb{R}} dy_3 (f_{13} g_{32} \pm g_{13} f_{32}), \end{aligned} \tag{5.4}$$

which satisfies the projective property (4.6).

i) Let us first investigate the time-independent symmetries of the equations generated by Φ_{12}.

a) Equation (4.33) yields

$$\hat{S}_{12}=D, \quad H_{12}=H_{12}(y_1,y_2), \tag{5.5a}$$

and starting operators $\hat{K}^0_{12}$ given by

$$\hat{N}_{12} \doteq q^-_{12}, \quad \hat{M}_{12} \doteq Dq^+_{12}+q^-_{12}D^{-1}q^-_{12}. \tag{5.5b}$$

b) The commutators of Φ_{12} with h_{12} imply the following operator equations:

$$[\Phi_{12},h_{12}]=4\alpha h'_{12}, \quad [\hat{N}_{12},h_{12}]=0, \quad [\hat{M}_{12},h_{12}]=2\alpha Dh'_{12}. \tag{5.6}$$

Hence, if

$$N^{(n)}_{12} \doteq \Phi^n_{12}\hat{N}_{12}\cdot 1, \quad M^{(n)}_{12} \doteq \Phi^n_{12}\hat{M}_{12}\cdot 1, \tag{5.7}$$

then Eq. (4.46) yields

$$\delta_{12}N^{(n)}_{12}=\sum_{\ell=1}^{n}(-4\alpha)^\ell\binom{n}{\ell}\Phi^{n-\ell}_{12}\hat{N}_{12}\delta^\ell_{12}, \tag{5.8a}$$

$$\delta_{12}M^{(n)}_{12}=\sum_{\ell=1}^{n}b_{n,\ell}\Phi^{n-\ell}_{12}\hat{M}_{12}\delta^\ell_{12}, \quad b_{n,\ell} \doteq (-4\alpha)^\ell\sum_{j=0}^{\ell}2^{-j}\binom{n-j}{\ell-j} \tag{5.8b}$$

(see Appendix B).

c) The Lie algebra of the starting symmetries is given by

$$\begin{aligned}&[\hat{N}_{12}H^{(1)}_{12},\hat{N}_{12}H^{(2)}_{12}]_d=-\hat{N}_{12}H^{(3)}_{12}, \quad [\hat{N}_{12}H^{(1)}_{12},\hat{M}_{12}H^{(2)}_{12}]_d=-\hat{M}_{12}H^{(3)}_{12},\\ &[\hat{M}_{12}H^{(1)}_{12},\hat{M}_{12}H^{(2)}_{12}]_d=-\Phi_{12}\hat{N}_{12}H^{(3)}_{12}, \quad H^{(3)}_{12} \doteq [H^{(1)}_{12},H^{(2)}_{12}]_I,\end{aligned} \tag{5.9}$$

where $[\,,\,]_d$, $[\,,\,]_I$ are defined by (4.3).

ii) We now investigate the Hamiltonian structure of the equations generated by Φ_{12}:

a) $\Phi_{12}\Theta^{(1)}_{12}=\Theta^{(1)}_{12}\Phi^*_{12}$, where

$$\Theta^{(1)}_{12}=D, \quad \Phi^*_{12}=D^2+q^+_{12}+D^{-1}q^+_{12}D+D^{-1}q^-_{12}D^{-1}q^-_{12}=D^{-1}\Phi_{12}D=\Psi_{12}.$$

We first note that both $\Theta^{(1)}_{12}=D$ and $\Theta^{(2)}_{12}=\Phi_{12}D$ are skew symmetric:

$$\Theta^{(1)*}_{12}=-D=-\Theta^{(1)}_{12}, \quad \Theta^{(2)*}_{12}=(\Phi_{12}D)^*=-D\Phi^*_{12}=-\Phi_{12}D=-\Theta^{(2)}_{12}.$$

Furthermore, the bracket

$$\begin{aligned}\{a_{12},b_{12},c_{12}\}&=\langle a_{12},\Theta^{(2)}_{12\,d}[\Theta^{(2)}_{12}b_{12}]c_{12}\rangle\\ &=\langle a_{12},(\Theta^{(2)}_{12}b_{12})^+D+D(\Theta^{(2)}_{12}b_{12})^++(\Theta^{(2)}_{12}b_{12})^-D^{-1}q^-_{12}+q^-_{12}D^{-1}(\Theta^{(2)}_{12}b_{12})^-)c_{12}\rangle\end{aligned}$$

satisfies the Jacobi identity. Also $\Theta^{(1)}_{12}$, $\Theta^{(2)}_{12}$ are compatible.

b) $\hat{\gamma}^0_{12}H_{12}=D^{-1}q^-_{12}H_{12}$ and $\hat{\gamma}^0_{12}=D^{-1}\hat{M}_{12}H_{12}$ are extended gradient functions. Thus the Theorems 4.1–4.4 imply:

Proposition 5.1. *Consider the two compatible Hamiltonian operators $\Theta^{(1)}_{12}=D$ and*

$$\Theta^{(2)}_{12}=D^3+q^+_{12}D+Dq^+_{12}+q^-_{12}D^{-1}q^-_{12},$$

and define

$$\Phi_{12} \doteq \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1} = D^2 + q_{12}^+ + Dq_{12}^+ D^{-1} + q_{12}^- D^{-1} q_{12}^- D^{-1},$$

$$\hat{N}_{12}^{(n)} \doteq \Phi_{12}^n \hat{N}_{12}, \quad \hat{M}_{12}^{(m)} \doteq \Phi_{12}^n \hat{M}_{12}, \quad \hat{\gamma}_{12}^{(n)} \doteq (\Theta_{12}^{(1)})^{-1} \hat{N}_{12} \quad \text{and/or} \quad (\Theta_{12}^{(1)})^{-1} \hat{M}_{12}^{(n)},$$

where the starting operator $\hat{N}_{12}$ *and* $\hat{M}_{12}$ *are defined by* $\hat{N}_{12} \doteq q_{12}^-$ *and* $\hat{M}_{12} = Dq_{12}^+ + q_{12}^- D^{-1} q_{12}^-$. *Then*

i) $M_{12}^{(m)} \doteq \hat{M}_{12}^{(m)} \cdot 1$ *and* $N_{12}^{(m)} \doteq \hat{N}_{12}^{(m)} \cdot 1$ *are extended symmetries for both classes of evolution equations*

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} N_{12}^{(n)} = N_{11}^{(n)}, \tag{5.10a}$$

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} M_{12}^{(n)} = M_{11}^{(n)}; \tag{5.10b}$$

namely

$$[M_{12}^{(m)}, \delta_{12} K_{12}^{(n)}]_d = [N_{12}^{(m)}, \delta_{12} K_{12}^{(n)}]_d = 0, \tag{5.11}$$

where $K_{12}^{(n)} = N_{12}^{(n)}$ *and/or* $M_{12}^{(n)}$.

ii) $\gamma_{12}^{(m)} \doteq \hat{\gamma}_{12}^{(m)} \cdot 1$ *are extended gradients of conserved quantities of both classes of evolution equations* (5.10), *namely*

$$\gamma_{12_d}^{(m)}[\delta_{12} K_{12}^{(n)}] + (\delta_{12} K_{12}^{(n)})_d^* [\gamma_{12}^{(m)}] = 0, \tag{5.12a}$$

$$(\hat{\gamma}_{12}^{(m)} H_{12})_d = (\hat{\gamma}_{12}^{(m)} H_{12})_d^*, \quad H_{12_x} = 0, \tag{5.12b}$$

*where * indicates the adjoint operation with respect to the bilinear form*

$$\langle f_{12}, g_{12} \rangle \doteq \int_{\mathbb{R}^3} dx dy_1 dy_2 f_{21} g_{12}. \tag{5.13}$$

iii) *The two classes of evolution equations* (5.10) *are bi-Hamiltonian, namely they can be written in the form*

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} \Theta_{12}^{(1)} \gamma_{12}^{(n)} = \int_{\mathbb{R}} dy_2 \delta_{12} \Theta_{12}^{(2)} \gamma_{12}^{(n-1)}. \tag{5.14}$$

iv) $M_{11}^{(m)}$ *and* $N_{11}^{(m)}$ *are infinitely many commuting symmetries of the classes of evolution equations* (5.10), *namely*

$$[M_{11}^{(m)}, M_{11}^{(n)}]_f = [M_{11}^{(m)}, N_{11}^{(n)}]_f = [N_{11}^{(m)}, N_{11}^{(n)}]_f = 0. \tag{5.15}$$

v) $\gamma_{11}^{(m)}$ *are infinitely many gradients of conserved quantities of the equations* (5.10), *namely*

$$\gamma_{11_f}^{(m)}[K_{11}^{(n)}] + K_{11_f}^{(n)+}[\gamma_{11}^{(m)}] = 0, \tag{5.16a}$$

$$\gamma_{11_f}^{(m)} = \gamma_{11_f}^{(m)+}, \tag{5.16b}$$

where $^+$ *indicates the operation of adjoint with respect to the bilinear form*

$$(f, g) \doteq \int_{\mathbb{R}^2} dx dy f g. \tag{5.17}$$

The corresponding conserved quantities are in involution with respect to the Poisson brackets

$$\{I^{(n)}, I^{(m)}\} \doteq \langle \delta_{12} \gamma_{12}^{(n)}, \Theta_{12} \gamma_{12}^{(m)} \rangle, \quad \Theta_{12} = \Theta_{12}^{(1)} \text{ or } \Theta_{12}^{(2)}; \tag{5.18a}$$

if

$$\Theta_{12}=\Theta_{12}^{(1)}, \quad \langle \delta_{12}\gamma_{12}^{(n)}, D\gamma_{12}^{(m)}\rangle = (\gamma_{11}^{(n)}, D\gamma_{12}^{(m)}). \tag{5.18b}$$

vi) *The equations* $M_{12}^{(m)}=0$ *and* $N_{12}^{(m)}=0$ *are Bäcklund Transformations for both classes of evolution equations* (5.10).

B. The 2×2 *AKNS Problem*

The spectral problem (1.8) yields the hereditary operator

$$\Phi_{12}=\sigma(P_{12}-Q_{12}^{+}P_{12}^{-}Q_{12}^{+}) \tag{5.19}$$

acting on off-diagonal matrices, where

$$Q_{12}^{\pm}F_{12} \doteqdot Q_1F_{12} \pm F_{12}Q_{12}, \tag{5.20a}$$

$$P_{12}F_{12} \doteqdot F_{12_x}-JF_{12_{y_1}}-F_{12_{y_2}}J. \tag{5.20b}$$

The integral representation of the basic operator $\hat{Q}_1 \doteqdot Q_1+JD_1$, implies an appropriate directional derivative:

$$\hat{Q}_1F_{12} \doteqdot (Q_1+JD_1)F_{12}=\int_{\mathbb{R}} dy_3Q_{13}F_{32}, \quad \hat{Q}_{1_d}[\sigma_{12}]F_{12}=\int_{\mathbb{R}} dy_3\sigma_{13}F_{32}, \tag{5.21}$$

and the adjoint of Eqs. (5.21) imply

$$\hat{Q}_1^{*}F_{12}=F_{12}Q_2-F_{12_{y_2}}J=\int_{\mathbb{R}} dy_3F_{13}Q_{32}, \quad \hat{Q}_{1_d}^{*}[\sigma_{12}]F_{12}=\int_{\mathbb{R}} dy_3F_{13}\sigma_{32}. \tag{5.22}$$

Then the reduction to the space of off-diagonal matrices performed in Sect. 3 induces the following derivative of the operator Φ_{12}:

$$\Phi_{12_d}[G_{12}]=-\sigma(G_{12}^{+}P_{12}^{-1}Q_{12}^{+}+Q_{12}^{+}P_{12}^{-1}G_{12}^{+}), \tag{5.23a}$$

$$G_{12}^{\pm}F_{12} \doteqdot \int_{\mathbb{R}} dy_3(G_{13}G_{32} \pm F_{13}G_{32}). \tag{5.23b}$$

Again the Leibnitz rule and property (4.6) are satisfied.

i) The investigation of the time-independent symmetries of the evolution equations generated by Φ_{12} gives the following results.

a) Equations (4.38) yield $\hat{S}_{12}=(Q_{12}^{+})^{-1}P_{12}$, the starting operators $\hat{K}_{12}^{0}$ are given by

$$\hat{N}_{12} \doteqdot Q_{12}^{-}, \quad \hat{M}_{12} \doteqdot Q_{12}^{-}\sigma, \tag{5.24}$$

and H_{12} is diagonal and such that $P_{12}H_{12}=0$.

b) The commutators of Φ_{12} with h_{12} imply the following operator equations:

$$[\Phi_{12}, h_{12}]=-2\alpha h_{12}', \quad [\hat{N}_{12}, h_{12}]=[\hat{M}_{12}, h_{12}]=0, \tag{5.25}$$

valid on arbitrary off-diagonal matrices. Hence, if

$$N_{12}^{(n)} \doteqdot \Phi_{12}^{n}\hat{N}_{12}\cdot I, \quad M_{12}^{(n)} \doteqdot \Phi_{12}^{n}\hat{M}_{12}\cdot I, \tag{5.26}$$

then Eq. (4.46) yields

$$\hat{\delta}_{12}N_{12}^{(n)} = \sum_{\ell=1}^{n} (2\alpha)^\ell \binom{n}{\ell} \Phi_{12}^{n-\ell}\hat{N}_{12}\hat{\delta}_{12}^\ell, \tag{5.27a}$$

$$\hat{\delta}_{12}M_{12}^{(n)} = \sum_{\ell=1}^{n} (2\alpha)^\ell \binom{n}{\ell} \Phi_{12}^{n-\ell}\hat{M}_{12}\hat{\delta}_{12}^\ell. \tag{5.27b}$$

c) The Lie algebra of the starting symmetries is given by

$$[\hat{N}_{12}H_{12}^{(1)}, \hat{N}_{12}H_{12}^{(2)}]_d = -\hat{N}_{12}H_{12}^{(3)}, \quad [\hat{N}_{12}H_{12}^{(1)}, \hat{M}_{12}H_{12}^{(2)}]_d = -\hat{M}_{12}H_{12}^{(3)},$$

$$[\hat{M}_{12}H_{12}^{(1)}, \hat{M}_{12}H_{12}^{(2)}]_d = -\hat{N}_{12}H_{12}^{(3)}, \quad H_{12}^{(3)} \doteq [H_{12}^{(1)}, H_{12}^{(2)}]_l. \tag{5.28}$$

ii) We now investigate the Hamiltonian structure of the equations generated by Φ_{12}:

a) $\Phi_{12}\Theta_{12}^{(1)} = \Theta_{12}^{(1)}\Phi_{12}^*$, where

$$\Theta_{12}^{(1)} = \sigma, \quad \Phi_{12}^* = \sigma(P_{12} - Q_{12}^- P_{12}^{-1} Q_{12}^-) = \sigma^{-1}\Phi_{12}\sigma = \Psi_{12}; \tag{5.29}$$

notice that on the space of off-diagonal matrices $\sigma F_{12} = \frac{1}{2}[\sigma, F_{12}]$, $\Theta_{12}^{(1)} = \sigma$ and $\Theta_{12}^{(2)} = \Phi_{12}\Theta_{12}^{(1)}$ are skew-symmetric in the space of off-diagonal matrices:

$$\langle F_{12}, \sigma G_{12}\rangle = -\langle \sigma F_{12}, G_{12}\rangle,$$

and

$$\Theta_{12}^{(2)*} = (\Phi_{12}\sigma)^* = -\sigma\Phi_{12}^* = -\Phi_{12}\sigma = -\Theta_{12}^{(2)}.$$

Furthermore, the bracket $\{A_{12}, B_{12}, C_{12}\} \doteq \langle A_{12}, \Theta_{12\,d}^{(2)}[\Theta_{12}^{(2)}B_{12}]C_{12}\rangle$ satisfies the Jacobi identity and $\Theta_{12}^{(1)}, \Theta_{12}^{(2)}$ are compatible.

b) $\hat{\gamma}_{12}^0 H_{12} = (\Theta_{12}^{(1)})^{-1}\hat{K}_{12}^0 H$ ($\hat{K}_{12}^0 = \hat{N}_{12}$ or $\hat{M}_{12}$) are extended gradients, thus Theorems 4.1–4.4 imply:

Proposition 5.2. *Consider the two compatible Hamiltonian operators $\Theta_{12}^{(1)} = \sigma$ and $\Theta_{12}^{(2)} = P_{12} - Q_{12}^- P_{12}^{-1} Q_{12}^-$ acting on off-diagonal matrices, and define*

$$\Phi_{12} \doteq \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1} = \sigma(P_{12} - Q_{12}^+ P_{12}^{-1} Q_{12}^+), \quad \hat{N}_{12}^{(n)} \doteq \Phi_{12}^n \hat{N}_{12},$$

$$\hat{M}_{12}^{(n)} \doteq \Phi_{12}^n \hat{M}_{12}, \quad \hat{\gamma}_{12}^{(n)} \doteq (\Theta_{12}^{(1)})^{-1}\hat{N}_{12}^{(n)} \quad \text{and/or} \quad (\Theta_{12}^{(1)})^{-1}\hat{M}_{12}^{(n)},$$

where the starting operators $\hat{N}_{12}$ and $\hat{M}_{12}$ are defined by $\hat{N}_{12} \doteq Q_{12}^-$ and $\hat{M}_{12} \doteq Q_{12}^-\sigma$. Then the results i)–vi) of Proposition 5.1 are all valid for the two classes of evolution equations

$$Q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} N_{12}^{(n)} = N_{11}^{(n)}, \tag{5.30a}$$

$$Q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} M_{12}^{(n)} = M_{11}^{(n)}, \tag{5.30b}$$

introducing trace in the right-hand side Eqs. (5.13) and (5.17) and replacing (5.18b) by

$$\Theta_{12} = \Theta_{12}^{(1)} = \sigma, \quad \langle \delta_{12}\gamma_{12}^{(n)}, \sigma\gamma_{12}^{(m)}\rangle = (\gamma_{11}^{(n)}, \sigma\gamma_{11}^{(m)}).$$

Appendix A

Now we show that the assumptions (4.30a), (4.30b) follow from (4.28a), without using the explicit form of the operator. We show this for the recursion operator associated with the Schrödinger eigenvalue problem.

Admissibility requires Φ_{12} to depend on $q_{12}^{\pm}$, moreover, (4.28a) and (3.13) imply that Φ_{12} depends linearly on q_{12}^{+}. Then, without loss of generality we have

$$\Phi_{12_d}[f_{12}]g_{12} = \sum_j c_j f_{12}^{+} d_j g_{12} + \sum_s p_s(q_{12}^{-}) f_{12}^{-} r_s(q_{12}^{-}) g_{12}, \tag{A.1a}$$

$$\Phi_{12_r}[f]g_{12} = \sum_j c_j(f_{11} + f_{22}) d_j g_{12} + \sum_s p_s(q_{12}^{-})(f_{11} - f_{22}) r_s(q_{12}^{-}) g_{12}, \tag{A.1b}$$

where c_i, d_i are arbitrary functions of D, D^{-1}; p_s, r_s are arbitrary functions of q_{12}^{-} and $f_{12}^{\pm}$ are defined in (5.4b).

Then the commutation property $[q_{12}^{-}, h_{12}] = 0$ implies

$$\Phi_{12_d}[h_{12} f_{12}]\delta_{12} g_{12} = h_{12}\Phi_{12_d}[f_{12}]\delta_{12} g_{12}, \tag{A.2a}$$

$$\Phi_{12_r}[f] h_{12} g_{12} = h_{12}\Phi_{12_r}[f] g_{12}. \tag{A.2b}$$

Appendix B

In this appendix we show that equations

$$[\Phi_{12}, h_{12}] = -\beta h'_{12}, \qquad h_{12} = h(y_1 - y_2), \tag{B.1a}$$

$$[\hat{K}_{12}^{0}, h_{12}] = -\tilde{\beta}\hat{S}_{12} h'_{12}, \tag{B.1b}$$

and some additional notions concerning the associated spectral problem, imply

$$\delta_{12} K_{12}^{(n)} = \sum_{\ell=0}^{n} b_{n,\ell}\Phi_{12}^{n-\ell}\hat{K}_{12}^{0}\delta_{12}^{\ell} \tag{B.2}$$

for suitable constants $b_{n,\ell}$.

We first observe that the case $\tilde{\beta} = 0$ is particularly simple; indeed, in this case

$$\delta_{12} K_{12}^{(n)} = \delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0} \cdot 1 = (\Phi_{12} + \beta\mathscr{D})^{n}\hat{K}_{12}^{0}\delta_{12} = \sum_{\ell=0}^{n} b_{n,\ell}\Phi_{12}^{n-\ell}\hat{K}_{12}^{0}\delta_{12}^{\ell}, \tag{B.3a}$$

$$b_{n,\ell} \doteq \beta^{\ell}\binom{n}{\ell}. \tag{B.3b}$$

This is the case for the two classes of evolution equations associated with the two-dimensional AKNS problem and for Eqs. (3.20). For the KP class (3.19), $\hat{K}_{12}^{0} = \hat{M}_{12} \doteq Dq_{12}^{+} + q_{12}^{-}D^{-1}q_{12}^{-}$, $\tilde{\beta} = \beta/2 = -2\alpha$, $\hat{S}_{12} = D$ and the result (B.2) is less straightforward.

In order to obtain it, we first show that

$$\Phi_{12}^{n}\Gamma_{12} \cdot 1 = 0, \qquad \forall n \geqq 0; \qquad \Gamma_{12} \doteq \Phi_{12} D - \hat{M}_{12}. \tag{B.4}$$

This result could be easily derived using the explicit form of Φ_{12} and $\hat{M}_{12}$. Here we give a different derivation using the underlying spectral problem (and the

consequent eigenvalue equation satisfied by Φ^{*}_{12}). This derivation is similar in spirit to the one of (B.1a) presented in Sect. 4.E.

From Eq. (4.38), it follows that Γ_{12} can be written as

$$\Gamma_{12}=\Delta_{12}D, \quad \Delta_{12}H_{12}\neq 0. \tag{B.5}$$

The operator Δ_{12}, which is part of Φ_{12}, is admissible depending on D, D^{-1}, $q^{\pm}_{12}$. If for any admissible operator L_{12}, we define $L^{(0)}_{12}$ as $L^{(0)}_{12}\doteq L_{12}|_{q=0}$, then

$$\Phi^{n}_{12}\Gamma_{12}\cdot 1=\Phi^{n}_{12}\Delta_{12}D\cdot 1=\Phi^{n}_{12}\Delta^{(0)}_{12}D\cdot 1=D\Psi^{n}_{12}\Delta^{(0)}_{12}\cdot 1, \tag{B.6}$$

since $D^{-1}qD\cdot 1=0$ and $[L^{(0)}_{12}, D]=0$. On the other hand, if $q=0$, $w=1$ solves Eq. (1.1) and its adjoint, then Eq. (1.7) implies that

$$\Psi^{(0)}_{12}\cdot 1=0 \quad (\text{and } \Delta^{(0)}_{12}\cdot 1=0). \tag{B.7}$$

Equations (B.7) imply $D\Psi^{n}_{12}\Delta^{(0)}_{12}\cdot 1=0$ which is equivalent to (B.4).

Equation (B.4) and Eqs. (B.1) imply (B.2). In fact, multiplying Eq. (B.4) by h_{12} and using Eqs. (B.1) we obtain

$$(\Phi_{12}+\beta\mathscr{D})^{n+1}D\cdot h_{12}=(\Phi_{12}+\beta\mathscr{D})^{n}(\hat{M}_{12}+\bar{\beta}\mathscr{D}D)\cdot h_{12}. \tag{B.8}$$

The above can be written in the following recursive way:

$$A_{n+1}(h_{12})=B_n(h_{12})+A_n(\bar{\beta}h'_{12}), \tag{B.9}$$

where

$$A_n(h_{12})\doteq\sum_{\ell=0}^{n}\beta^{\ell}\binom{n}{\ell}\Phi^{n-\ell}_{12}D\cdot h^{(\ell)}_{12}, \quad A_0(h_{12})=0, \tag{B.10a}$$

$$B_n(h_{12})\doteq\sum_{\ell=0}^{n}\beta^{\ell}\binom{n}{\ell}\Phi^{n-\ell}_{12}\hat{M}_{12}h^{(\ell)}_{12}, \quad B_0(h_{12})=\hat{M}_{12}h_{12}, \tag{B.10b}$$

$$h^{(\ell)}_{12}\doteq\frac{\partial^{\ell}h_{12}}{\partial y_1^{\ell}}. \tag{B.10c}$$

The solution $A_{n+1}(h_{12})=\sum_{s=0}^{n}B_{n-s}(\bar{\beta}^{s}h^{(s)}_{12})$ of Eqs. (B.9) and (B.10) implies Eq. (B.2). Indeed,

$$\begin{aligned}\delta_{12}K^{(n)}_{12}&=\delta_{12}\Phi^{n}_{12}\hat{M}_{12}\cdot 1=\delta_{12}\Phi^{n+1}_{12}D\cdot 1=A_{n+1}(\delta_{12})\\&=\sum_{s=0}^{n}B_{n-s}(\bar{\beta}^{s}\delta^{s}_{12})=\sum_{\ell=0}^{n}b_{n,\ell}\Phi^{n-\ell}_{12}\hat{M}_{12}\delta^{\ell}_{12},\end{aligned} \tag{B.11}$$

where

$$b_{n,\ell}\doteq\sum_{s=0}^{\ell}\beta^{\ell-s}\bar{\beta}^{s}\binom{n-s}{\ell-s}. \tag{B.12}$$

For example, for the KP equation ($\hat{M}=Dq^{+}_{12}+q^{-}_{12}D^{-1}q^{-}_{12}$):

$$\delta_{12}M^{(1)}_{12}=\delta_{12}\Phi_{12}\hat{M}_{12}\cdot 1=\Phi_{12}\hat{M}_{12}\delta_{12}-6\alpha\hat{M}_{12}\delta'_{12}, \tag{B.13a}$$

and for the DS equation ($\hat{M}_{12}=Q^{-}_{12}\sigma$):

$$\delta_{12}M^{(2)}_{12}=\delta_{12}\Phi^{2}_{12}\hat{M}_{12}\cdot I=\Phi^{2}_{12}\hat{M}_{12}\delta_{12}+4\alpha\Phi_{12}\hat{M}_{12}\delta'_{12}+4\alpha^{2}\hat{M}_{12}\delta^{2}_{12}. \tag{B.13b}$$

Finally, we use again Eq. (B.4) to derive the following interesting equation:

$$\Phi^{n+1}_{12}D\cdot h_{12}=\sum_{s=0}^{n}(\bar{\beta}-\beta)^{s}\Phi^{n-s}_{12}\hat{M}_{12}h^{(s)}_{12}. \tag{B.14}$$

Multiplying Eq. (B.4) by h_{12} and using (B.1a), we obtain

$$\Phi^{j}_{12}h_{12}(\Phi^{n-j+1}_{12}D\cdot 1-\Phi^{n-j}_{12}\hat{M}_{12}\cdot 1)=0, \qquad j\leqq n. \tag{B.15}$$

Equation (B.15) for $j=n$ and Eqs. (B.1) imply

$$\Phi^{n+1}_{12}D\cdot h_{12}=\Phi^{n}_{12}\hat{M}_{12}\cdot h_{12}+(\bar{\beta}-\beta)\Phi^{n}_{12}D\cdot h^{(1)}_{12}, \tag{B.16}$$

and hence Eq. (B.14).

Remark B.1. i) Equation (B.14) contains (B.4) if $h_2=1$.

ii) Equation (B.14) can be used to obtain (B.2), (B.12) in an alternative way. In fact,

$$\begin{aligned}\delta_{12}M^{(n)}_{12}=\delta_{12}\Phi^{n+1}_{12}D\cdot 1&=\sum_{r=0}^{n}\beta^{r}\binom{n+1}{r}\Phi^{n+1-r}_{12}D\cdot h^{(r)}_{12}\\&=\sum_{\ell=0}^{n}\left[\sum_{s=0}^{\ell}\beta^{\ell-s}\binom{n+1}{\ell-s}(\bar{\beta}-\beta)^{s}\right]\Phi^{n-\ell}_{12}\hat{M}_{12}h^{(\ell)}_{12}=\sum_{\ell=0}^{n}b_{n,\ell}\Phi^{n-\ell}_{12}\hat{M}_{12}\cdot h^{(\ell)}_{12},\end{aligned}$$

since the identity

$$\binom{n-s}{\ell-s}=\sum_{v=s}^{\ell}(-1)^{v-s}\binom{v}{s}\binom{n+1}{\ell-v}, \qquad s\leqq\ell\leqq n, \tag{B.17}$$

implies that

$$\sum_{s=0}^{\ell}\beta^{\ell-s}(\bar{\beta}-\beta)^{s}\binom{n+1}{\ell-s}=\sum_{s=0}^{\ell}\beta^{\ell-s}\bar{\beta}^{s}\binom{n-s}{\ell-s}, \qquad \ell\leqq n.$$

Appendix C

In this appendix we define explicitly the directional derivative introduced in Sect. 4 for the KP and DS classes. Then we use it to verify some of the results contained in this paper.

C1. Evolution Equations Associated with the KP Equation

The directional derivative of the basic operators $q^{\pm}_{12}\doteq q_1\pm q_2+\alpha(D_1\mp D_2)$ associated with the non-stationary Schrödinger problem (1.1) is the usual Frechét

derivative with respect to the kernel q_{12} of their integral representation:

$$q^{\pm}_{12}g_{12} = \int_{\mathbb{R}} dy_3(q_{13}g_{32} \pm g_{13}q_{32}), \quad q_{12} = \delta_{12}q_1 + \alpha\delta'_{12}, \tag{C.1a}$$

$$q^{\pm}_{12_d}[f_{12}]g_{12} = f^{\pm}_{12}g_{12}, \tag{C.1b}$$

$$f^{\pm}_{12}g_{12} \doteq \int_{\mathbb{R}} dy_3(f_{13}g_{32} \pm g_{13}f_{32}). \tag{C.1c}$$

In order to make explicit calculations, it is convenient to use the following basic identities of this algebra of integral operators

$$a^{\pm}_{12}b_{12} = \pm b^{\pm}_{12}a_{12}, \tag{C.2a$_\pm$}$$

$$(a^{\pm}_{12}b^{\pm}_{12} - b^{\pm}_{12}a^{\pm}_{12})c_{12} = (a^{-}_{12}b_{12})^{-}c_{12} = -c^{-}_{12}a^{-}_{12}b_{12}, \tag{C.2b$_\pm$}$$

$$(a^{+}_{12}b^{-}_{12} \mp b^{\mp}_{12}a^{\pm}_{12})c_{12} = (a_{12} \mp b_{12})^{\pm}c_{12} = \pm c^{\pm}_{12}a^{\mp}_{12}b_{12}; \tag{C.2c$_\pm$}$$

where a_{12}, b_{12}, c_{12} are arbitrary functions of x, y_1, y_2 decaying at ∞ and $a^{\pm}_{12}, b^{\pm}_{12}, c^{\pm}_{12}$ are the corresponding integral operators defined in (C.1c).

The integral representations (C.1a) imply that the basic operators $q^{\pm}_{12}$ can replace $a^{\pm}_{12}$ (and/or $b^{\pm}_{12}, c^{\pm}_{12}$) in Eqs. (C.2). For instance, if $a^{\pm}_{12} = f^{\pm}_{12}$, $b^{\pm}_{12} = q^{\pm}_{12}$, and $c^{\pm}_{12} = H^{\pm}_{12}$, the identity (C.2c)$_-$ becomes

$$f^{\pm}_{12}q^{-}_{12}H_{12} + q^{+}_{12}f^{-}_{12}H_{12} + H^{-}_{12}q^{+}_{12}f_{12} = 0, \tag{C.3}$$

where we have also used Eq. (C.2a)$_+$ to replace $f^{+}_{12}q_{12}$ by the expression $q^{+}_{12}f_{12}$ in which the kernel q_{12} does not appear explicitly.

It is worthwhile to remark that formulas (C.2) can also be interpreted as matrix identities in which a, b, c are matrices and the $\pm$ operations denote anticommutator and commutator:

$$a^{\pm}b \doteq ab \pm ba. \tag{C.4}$$

Interpreting the operation $a^{\pm}_{12}b_{12}$ as in (C.4), the recursion operator (1.2) of the KP class becomes the recursion operator

$$\Phi = D^2 + q^{+} + Dq^{+}D^{-1} + q^{-}D^{-1}q^{-}D^{-1} \tag{C.5}$$

associated with the $N \times N$ matrix Schrödinger problem in 1 dimension and introduced by Calogero and Degasperis [38]. Then important properties of the recursion operator of the KP, like its hereditariness (4.21), are equivalent to the corresponding properties of the matrix operator (C.5)! This important connection is explained from the fact that the 2 + 1 dimensional systems considered here can be viewed as reductions of certain evolution equations nonlocal in y. These equations are directly connected to matrix evolution equations (see Sect. 5 of [35]).

Now we use Eqs. (C.2) to verify some results concerning the symmetries and the bi-Hamiltonian structure of Eqs. (3.19) and (3.20).

a) Φ_{12} is a strong symmetry of $\hat{N}_{12}H_{12}$, where $\hat{N}_{12} = q^{-}_{12}$ and $H_{12_x} = 0$ (this result is a consequence of Lemma 4.2; but here it is verified directly).

$$
\begin{aligned}
&\Phi_{12_d}[q_{12}^- H_{12}] f_{12} - (q_{12}^- H_{12})_d[\Phi_{12} f_{12}] + \Phi_{12}(q_{12}^- H_{12})_d[f_{12}] \\
&\quad = (q_{12}^- H_{12})^+ f_{12} + D(q_{12}^- H_{12})^+ D^{-1} f_{12} \\
&\qquad + (q_{12}^- H_{12})^- D^{-1} q_{12}^- D^{-1} f_{12} + q_{12}^- D^{-1}(q_{12}^- H_{12})^- D^{-1} f_{12} \\
&\qquad - (D^2 f_{12} + q_{12}^+ f_{12} + D q_{12}^+ D^{-1} f_{12} + q_{12}^- D^{-1} q_{12}^- D^{-1} f_{12})^- H_{12} \\
&\qquad + (D^2 + q_{12}^+ + D q_{12}^+ D^{-1} + q_{12}^- D^{-1} q_{12}^- D^{-1}) f_{12}^- H_{12} = 0, \quad \text{since:}
\end{aligned}
$$

the terms without $q_{12}^{\pm}$ give

$$
-f_{12_{xx}}^- H_{12} + D^2 f_{12}^- H_{12} = 0;
$$

the terms linear in $q_{12}^{\pm}$ give

$$
\begin{aligned}
&(q_{12}^- H_{12})^+ f_{12} + D(q_{12}^- H_{12})^+ D^{-1} f_{12} - (q_{12}^+ f_{12})^- H_{12} - D(q_{12}^+ D^{-1} f_{12})^- H_{12} \\
&\quad + q_{12}^+ f_{12}^- H_{12} + D q_{12}^+ D^{-1} f_{12}^- H_{12} = f_{12}^+ q_{12}^- H_{12} + q_{12}^+ f_{12}^- H_{12} + H_{12}^- q_{12}^+ f_{12} \\
&\qquad D((D^{-1} f_{12})^+ q_{12}^- H_{12} + q_{12}^+ (D^{-1} f_{12})^- H_{12} + H_{12}^- q_{12}^+ D^{-1} f_{12}) = 0,
\end{aligned}
$$

using Eq. (C.3);

the terms quadratic in $q_{12}^{\pm}$ give

$$
\begin{aligned}
&(q_{12}^- H_{12})^- D^{-1} q_{12}^- D^{-1} f_{12} + H_{12}^- q_{12}^- D^{-1} q_{12}^- D^{-1} f_{12} \\
&\quad + q_{12}^- D^{-1}(-(D^{-1} f_{12})^- q_{12}^- H_{12} + q_{12}^- D^{-1} f_{12}^- H_{12}) \\
&\quad = (-q_{12}^- H_{12}^- + H_{12}^- q_{12}^- + (q_{12}^- H_{12})^-) D^{-1} q_{12}^- D^{-1} f_{12} = 0.
\end{aligned}
$$

b) The Lie algebra of the starting symmetries is given by the following equations:

$$
\begin{aligned}
&[\hat N_{12} H_{12}^{(1)}, \hat N_{12} H_{12}^{(2)}]_d = -\hat N_{12} H_{12}^{(3)}, \qquad [\hat N_{12} H_{12}^{(1)}, \hat M_{12} H_{12}^{(2)}]_d = -\hat M_{12} H_{12}^{(3)}, \\
&[\hat M_{12} H_{12}^{(1)}, \hat M_{12} H_{12}^{(2)}]_d = -\Phi_{12} \hat N_{12} H_{12}^{(3)}, \qquad H_{12}^{(3)} \doteq [H_{12}^{(1)}, H_{12}^{(2)}]_l = (H_{12}^{(1)})^- H_{12}^{(2)},
\end{aligned}
\tag{C.6}
$$

where

$$
\hat N_{12} \doteq q_{12}^-, \qquad \hat M_{12} \doteq D q_{12}^+ + q_{12}^- D^{-1} q_{12}^-, \qquad H_{12_x} = 0.
$$

Equation (C.6a) holds, since,

$$
\begin{aligned}
[q_{12}^- H_{12}^{(1)}, q_{12}^- H_{12}^{(2)}]_d &= (q_{12}^- H_{12}^{(2)})^- H_{12}^{(1)} - (q_{12}^- H_{12}^{(1)})^- H_{12}^{(2)} \\
&= -(H_{12}^{(1)})^- q_{12}^- H_{12}^{(2)} + (H_{12}^{(2)})^- q_{12}^- H_{12}^{(1)} = -q_{12}^- (H_{12}^{(1)})^- H_{12}^{(2)},
\end{aligned}
$$

using (C.2b)$_-$. Equation (C.6b) holds since:

$$
\begin{aligned}
&[q_{12}^- H_{12}^{(1)}, (D q_{12}^+ + q_{12}^- D^{-1} q_{12}^-) H_{12}^{(2)}]_d \\
&\quad = ((D q_{12}^+ + q_{12}^- D^{-1} q_{12}^-) H_{12}^{(2)})^- H_{12}^{(1)} - D(q_{12}^- H_{12}^{(1)})^+ H_{12}^{(2)} \\
&\qquad - (q_{12}^- H_{12}^{(1)})^- D^{-1} q_{12}^- H_{12}^{(2)} - q_{12}^- D^{-1}(q_{12}^- H_{12}^{(1)})^- H_{12}^{(2)} \\
&\quad = -D((H_{12}^{(1)})^- q_{12}^+ H_{12}^{(2)} + (H_{12}^{(2)})^+ q_{12}^- H_{12}^{(1)}) - (H_{12}^{(1)})^- q_{12}^- D^{-1} q_{12}^- H_{12}^{(2)} \\
&\qquad + (D^{-1} q_{12}^- H_{12}^{(2)})^- q_{12}^- H_{12}^{(1)} + q_{12}^- D^{-1}(H_{12}^{(2)})^- q_{12}^- H_{12}^{(1)} \\
&\quad = -D q_{12}^+ (H_{12}^{(1)})^- H_{12}^{(2)} + q_{12}^- D^{-1}(-(H_{12}^{(1)})^- q_{12}^- H_{12}^{(2)} + (H_{12}^{(2)})^- q_{12}^- H_{12}^{(1)}) \\
&\quad = -\hat M_{12} (H_{12}^{(1)})^- H_{12}^{(2)}.
\end{aligned}
$$

The verification of Eq. (C.6c) is left to the reader.

The notion of an extended symmetry σ_{12} of the evolution equation $q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} K_{12}^{(n)} = K_{11}^{(n)}$ plays an important role in $2+1$ dimensions. σ_{12} is a solution of the equation

$$\sigma_{12_f}[K^{(n)}] = (\delta_{12} K_{12}^{(n)})_d [\sigma_{12}], \tag{C.7a}$$

where

$$(\delta_{12} K_{12}^{(n)})_d \doteq \sum_{\ell=0}^{n} b_{n,\ell} (\Phi_{12}^{n-\ell} \hat{R}_{12}^{0} \delta_{12}^{\ell})_d . \tag{C.7b}$$

Again the use of Eqs. (C.2) and the property

$$(\delta_{12}^{n})^{\pm} f_{12} = (D_1^n \pm (-1)^n D_2^n) f_{12} \tag{C.8}$$

simplify the calculations of the operator (C.7b).

c) σ_{12} is an extended symmetry of

i) the wave equation $q_{1_t} = M_{11}^{(0)} = 2q_{1_x}$ iff

$$\sigma_{12_f}[2q_x] = 2D\sigma_{12}; \tag{C.9a}$$

ii) the KP equation $q_{1_t} = M_{11}^{(1)} = 2(q_{1_{xxx}} + 6q_1 q_{1_x} + 3\alpha^2 D^{-1} q_{1_{y_1 y_1}})$ iff

$$\begin{aligned}\sigma_{12_f}[2(q_{xxx} + 6qq_x + 3\alpha^2 D^{-1} q_{yy})] = {} & 2[D^3 + 6D(q_1 + q_2) - 3\alpha(D^{-1}(q_{1_{y_1}} - q_{2_{y_2}})) \\ & + 6\alpha(q_1 - q_2)D^{-1}(D_1 + D_2) + 6\alpha D^{-1}(D_1 + D_2)^2]\sigma_{12} .\end{aligned} \tag{C.9b}$$

$$\begin{aligned}(\delta_{12} K_{12}^{(0)})_d[f_{12}] = (\hat{M}_{12}\delta_{12})_d[f_{12}] = {} & Df_{12}^{+}\delta_{12} + f_{12}^{-} D^{-1} q_{12}^{-} \delta_{12} \\ & + q_{12}^{-} D^{-1} f_{12}^{-} \delta_{12} = 2Df_{12} .\end{aligned}$$

$$\begin{aligned}(\delta_{12} K_{12}^{(1)})_d[f_{12}] &= (\Phi_{12}\hat{M}_{12}\delta_{12} - 6\alpha \hat{M}_{12}\delta'_{12})_d[f_{12}] \\ &= \Phi_{12_d}[f_{12}]\hat{M}_{12}\delta_{12} + \Phi_{12}(\hat{M}_{12}\delta_{12})_d[f_{12}] - 6\alpha(\hat{M}_{12}\delta'_{12})_d[f_{12}] \\ &= (f_{12}^{+} + Df_{12}^{+}D^{-1} + f_{12}^{-}D^{-1}q_{12}^{-}D^{-1} + q_{12}^{-}D^{-1}f_{12}^{-}D^{-1})(Dq_{12}^{+} + q_{12}^{-}D^{-1}q_{12}^{-})\delta_{12} \\ &\quad + D^2 + q_{12}^{+} + Dq_{12}^{+}D^{-1} + q_{12}^{-}D^{-1}q_{12}^{-}D^{-1})(Df_{12}^{+} + f_{12}^{-}D^{-1}q_{12}^{-} + q_{12}^{-}D^{-1}f_{12}^{-})\delta_{12} \\ &\quad - 6\alpha(Df_{12}^{+} + f_{12}^{-}D^{-1}q_{12}^{-} + q_{12}^{-}D^{-1}f_{12}^{-})\delta'_{12} \\ &= 2[D^3 + 6D(q_1 + q_2) - 3\alpha(D^{-1}(q_{1_{y_1}} - q_{2_{y_2}})) \\ &\quad + 6\alpha(q_1 - q_2)D^{-1}(D_1 + D_2) + 6\alpha^2 D^{-1}(D_1 + D_2)^2] ,\end{aligned}$$

since, for instance:

$$\begin{aligned}f_{12}^{+}Dq_{12}^{+}\delta_{12} &= (Dq_{12})^{+}f_{12}^{+}\delta_{12} - \delta_{12}^{-}f_{12}^{-}q_{12} = 2(q_1 + q_2)_x f_{12} , \\ Df_{12}^{+}q_{12}^{+}\delta_{12} &= 2Df_{12}^{+}q_{12} = 2Dq_{12}^{+}f_{12} , \\ Df_{12}^{+}\delta'_{12} &= D(\delta'_{12})^{+}f_{12} = D(D_1 - D_2)f_{12} , \\ f_{12}^{-}D^{-1}q_{12}^{-}\delta'_{12} &= -(D^{-1}q_{12}^{-}\delta'_{12})^{-}f_{12} = (D^{-1}(\delta'_{12})^{-}q_{12})^{-}f_{12} \\ &= (D^{-1}(D_1 + D_2)q_{12})^{-}f_{12} = (D^{-1}(q_{1_{y_1}} - q_{2_{y_2}}))f_{12} , \\ q_{12}^{-}D^{-1}f_{12}^{-}\delta'_{12} &= -q_{12}^{-}D^{-1}(\delta'_{12})^{-}f_{12} = -q_{12}^{-}D^{-1}(D_1 + D_2)f_{12} ,\end{aligned}$$

and we have used, for the first and only time in this appendix, the explicit representation (C.1a) of q_{12}.

In order to investigate the Hamiltonian structure of the equations generated by Φ_{12}, in addition to Eqs. (C.2) we use the following properties:

$$a_{12}^{\pm *} = \pm a_{12}^{\pm}, \qquad q_{12}^{\pm *} = \pm q_{12}^{\pm}. \tag{C.10}$$

These properties follow from the definitions (C.1c), (C.1a), and (4.8):

$$\begin{aligned}\langle f_{12}, a_{12}^{\pm} g_{12}\rangle &= \int_{\mathbb{R}^4} dx dy_1 dy_2 dy_3 f_{21}(a_{13}g_{32} \pm g_{13}a_{32}) \\ &= \int_{\mathbb{R}^4} dx dy_1 dy_2 dy_3 (f_{23}a_{31} \pm f_{31}a_{23}) g_{12} \\ &= \pm \langle a_{12}^{\pm} f_{12}, g_{12}\rangle .\end{aligned}$$

d) $\gamma_{12}^0 H_{12} = D^{-1}\hat{K}_{12}^0 H_{12}$ ($\hat{K}_{12}^0 = \hat{N}_{12}$ and $\hat{M}_{12}$) are extended gradients, namely $(\gamma_{12}^0 H_{12})_d^* = (\gamma_{12}^0 H_{12})_d$.

i) If $\hat{K}_{12}^0 = \hat{N}_{12}$, then $(\gamma_{12}^0 H_{12})_d[g_{12}] = D^{-1} g_{12}^- H_{12}$ and

$$\begin{aligned}\langle f_{12}, (\gamma_{12}^0 H_{12})_d[g_{12}]\rangle &= \langle f_{12}, D^{-1} g_{12}^- H_{12}\rangle = \langle D^{-1} f_{12}, H_{12}^- g_{12}\rangle \\ &= -\langle H_{12}^- D^{-1} f_{12}, g_{12}\rangle = \langle D^{-1} f_{12}^- H_{12}, g_{12}\rangle \\ &= \langle (\gamma_{12}^0 H_{12})_d[f_{12}], g_{12}\rangle .\end{aligned}$$

ii) If $\hat{K}_{12}^0 = \hat{M}_{12}$, then

$$(\gamma_{12}^0 H_{12})_d[g_{12}] = (g_{12}^+ + D^{-1} g_{12}^- D^{-1} g_{12}^- + D^{-1} q_{12}^- D^{-1} g_{12}^-) H_{12}$$

and

$$\begin{aligned}\langle f_{12}, (\gamma_{12}^0 H_{12})_d[g_{12}]\rangle &= \langle f_{12}, g_{12}^+ H_{12} + D^{-1} g_{12}^- D^{-1} q_{12}^- H_{12} + D^{-1} q_{12} D^{-1} g_{12}^- H_{12}\rangle \\ &= \langle f_{12}, (H_{12}^+ - D^{-1}((D^{-1} q_{12}^- H_{12})^- + q_{12}^- D^{-1} H_{12}^-)) g_{12}\rangle \\ &= \langle (H_{12}^+ - [(D^{-1} q_{12}^- H_{12})^- + H_{12}^- D^{-1} q_{12}^-] D^{-1}) f_{12}, g_{12}\rangle \\ &= \langle (H_{12}^+ - D^{-1}((D^{-1} q_{12}^- H_{12})^- + q_{12}^- H_{12}^- D^{-1})) f_{12}, g_{12}\rangle \\ &= \langle (\gamma_{12}^0 H_{12})_d[f_{12}], g_{12}\rangle .\end{aligned}$$

e) In [35] we show that

$$\gamma_{12}^{(n)} = \mathrm{grad}_{12} I_n, \tag{C.11a}$$

$$\begin{aligned} I_n &\doteq \frac{1}{2(2n+3)} \langle \gamma_{12}^{(n+1)}, \delta_{12}\rangle = \frac{1}{2(2n+3)} \int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} \gamma_{12}^{(n+1)} \\ &= \frac{1}{2(2n+3)} \int_{\mathbb{R}^2} dx dy_1 \gamma_{11}^{(n+1)}, \end{aligned} \tag{C.11b}$$

where $\gamma_{12}^{(n)} \doteq D^{-1} K_{12}^{(n)}$ and $\hat{K}_{12}^0 = \hat{M}_{12}$. Here we directly verify this result for $n=0$,

$$\begin{aligned} I_{0_d}[f_{12}] &= \tfrac{1}{6}\langle \delta_{12}, \gamma_{12_d}^{(1)}[f_{12}]\rangle \\ &= \tfrac{1}{6}\langle \gamma_{12_d}^{(1)*}[\delta_{12}], f_{12}\rangle = \tfrac{1}{6}\langle \gamma_{12_d}^{(1)}[\delta_{12}], f_{12}\rangle \\ &= \tfrac{1}{6}\langle \Phi_{12_d}^*[\delta_{12}]\gamma_{12}^{(0)} + \Phi_{12}^* \gamma_{12_d}^{(0)}[\delta_{12}] \cdot 1, f_{12}\rangle \\ &= \tfrac{1}{6}\langle 4\gamma_{12}^{(0)} + 2\Phi_{12}^* \cdot 1, f_{12}\rangle = \langle \gamma_{12}^{(0)}, f_{12}\rangle , \end{aligned} \tag{C.12}$$

which implies that $\gamma_{12}^{(0)} = \mathrm{grad}_{12} I_0$. (In this derivation we have used the property $\gamma_{12_d}^{(1)*} = \gamma_{12_d}^{(1)}$.)

f) The bracket $\{a_{12}, b_{12}, c_{12}\} \doteq \langle a_{12}, \Theta_{12_d}^{(2)}[\Theta_{12}^{(2)} b_{12}] c_{12}\rangle$, $\Theta_{12}^{(2)} \doteq \Phi_{12} D$ satisfies the Jacobi identity for every a_{12}, b_{12}, c_{12}. Here we only display some of the calculations for the linear terms in $q_{12}^{\pm}$.

$$\begin{aligned}
&\langle a_{12}, [(q_{12}^{+} Db_{12} + Dq_{12}^{+} b_{12})^{+} D + D(q_{12}^{+} Db_{12} + Dq_{12}^{+} b_{12})^{+} \\
&\quad + (D^3 b_{12})^{-} D^{-1} q_{12}^{-} + q_{12}^{-} D^{-1} (D^3 b_{12})^{-}] c_{12} \\
&\quad + \text{cyclic permutations of } a_{12}, b_{12}, c_{12} \\
&= \{a_{12}, b_{12}, c_{12}\} + \langle [D(q_{12}^{+} Dc_{12} + Dq_{12}^{+} c_{12})^{+} + (q_{12}^{+} Dc_{12} + Dq_{12}^{+} c_{12})^{+} D \\
&\quad - q_{12}^{-} D^{-1} (D^3 c_{12})^{-} - (D^3 c_{12})^{-} D^{-1} q_{12}^{-}] b_{12}, a_{12}\rangle \\
&\quad + \langle c_{12}, (Db_{12})^{+} (q_{12}^{+} Da_{12} + Dq_{12}^{+} a_{12}) + Db_{12}^{+} (q_{12}^{+} Da_{12} + Dq_{12}^{+} a_{12}) \\
&\quad - (D^{-1} q_{12}^{-} b_{12})^{-} D^3 a_{12} - q_{12}^{-} D^{-1} b_{12}^{-} D^3 a_{12}\rangle = \langle a_{12}, L_{12}(b_{12}, c_{12})\rangle ,
\end{aligned}$$

where

$$\begin{aligned}
L_{12}(b_{12}, c_{12}) \doteq{} & (q_{12}^{+} Db_{12} + Dq_{12}^{+} b_{12}) Dc_{12} + D(q_{12}^{+} Db_{12} + Dq_{12}^{+} b_{12})^{+} c_{12} \\
& + (D^3 b_{12})^{-} D^{-1} q_{12}^{-} c_{12} + q_{12}^{-} D^{-1} (D^3 b_{12})^{-} c_{12} \\
& + D(q_{12}^{+} Dc_{12} + Dq_{12}^{+} c_{12})^{+} b_{12} + (q_{12}^{+} Dc_{12} + Dq_{12}^{+} c_{12})^{+} Db_{12} \\
& - q_{12}^{-} D^{-1} (D^3 c_{12})^{-} b_{12} - (D^3 c_{12})^{-} D^{-1} q_{12}^{-} b_{12} - Dq_{12}^{+} (Db_{12})^{+} c_{12} \\
& - q_{12}^{+} D (Db_{12})^{+} c_{12} + Dq_{12}^{+} b_{12}^{+} Dc_{12} + q_{12}^{+} Db_{12}^{+} Dc_{12} \\
& - D^3 (D^{-1} q_{12}^{-} b_{12})^{-} c_{12} - D^3 b_{12}^{-} D^{-1} q_{12}^{-} c_{12} .
\end{aligned}$$

Using Eqs. (C.2), it is possible to show that $L_{12}(b_{12}, c_{12}) = 0$, $\forall b_{12}, c_{12}$.

C2. Evolution Equations Associated with the DS Equation

As in the previous case, it is easy to check from their definitions

$$Q_{12}^{\pm} G_{12} \doteq Q_1 G_{12} \pm G_{12} Q_2 = \int_{\mathbb{R}} dy_3 (Q_{13} G_{32} \pm G_{13} Q_{32}), \qquad Q_{12} = \delta_{12} Q_1 , \tag{C.13a}$$

$$Q_{12_d}^{\pm}[F_{12}] G_{12} = F_{12}^{\pm} G_{12} , \tag{C.13b}$$

$$F_{12}^{\pm} G_{12} \doteq \int_{\mathbb{R}} dy_3 (F_{13} G_{32} \pm G_{13} F_{32}) , \tag{C.13c}$$

that the operators $Q_{12}^{\pm}$ and $F_{12}^{\pm}$ satisfy Eqs. (C.2) and (C.10). Moreover, it is possible to show that the operator P_{12}, defined by

$$P_{12} F_{12} \doteq F_{12_x} - J F_{12_{y_1}} - F_{12_{y_2}} J , \tag{C.14}$$

satisfies the following equations

$$P_{12} F_{12}^{\pm} G_{12} = (P_{12} F_{12})^{\pm} G_{12} + F_{12}^{\pm} P_{12} G_{12} , \tag{C.15a}$$

$$\begin{aligned}
P_{12}^{-1} F_{12}^{\pm} G_{12} &= (P_{12}^{-1} F_{12})^{\pm} G_{12} - P_{12}^{-1} (P_{12}^{-1} F_{12})^{\pm} P_{12} G_{12} \\
&= F_{12}^{\pm} P_{12}^{-1} G_{12} - P_{12}^{-1} (P_{12} F_{12})^{\pm} P_{12}^{-1} G_{12} .
\end{aligned} \tag{C.15b}$$

Now we use Eqs. (C.13), (C.2), and (C.15) to verify some result concerning symmetries and bi-Hamiltonian structure of Eqs. (3.35) and (3.38).

a) Φ_{12} is a strong symmetry for $\hat{K}^0_{12}H_{12}$, where $\hat{K}^0_{12}=\hat{N}_{12}\doteq Q^-_{12}$ and $P_{12}H_{12}=0$, H_{12} diagonal.

$$\begin{aligned}&\Phi_{12_d}[Q^-_{12}H_{12}]F_{12}-(Q^-_{12}H_{12})_d[\Phi_{12}F_{12}]+\Phi_{12}(Q^-_{12}H_{12})_d[F_{12}]\\&\quad=-\sigma[(Q^-_{12}H_{12})^+P^{-1}_{12}Q^+_{12}+Q^+_{12}P^{-1}_{12}(Q^-_{12}H_{12})^+]F_{12}\\&\qquad-(\sigma(P_{12}-Q^+_{12}P^{-1}_{12}Q^+_{12})F_{12})^-H_{12}+\sigma(P_{12}-Q^+_{12}P^{-1}_{12}Q^+_{12})F^-_{12}H_{12}=0,\quad\text{since:}\end{aligned}$$

the terms without $Q^{\pm}_{12}$ give

$$-\sigma(P_{12}F_{12})^-H_{12}+\sigma P_{12}F^-_{12}H_{12}=0;$$

the terms with $Q^{\pm}_{12}$ give

$$\begin{aligned}&-\sigma[((Q^-_{12}H_{12})^++H^-_{12}Q^+_{12})P^{-1}_{12}Q^+_{12}F_{12}+Q^+_{12}P^{-1}_{12}(F^+_{12}Q^-_{12}H_{12}-Q^+_{12}F^-_{12}H_{12})]\\&\quad=-\sigma Q^+_{12}P^{-1}_{12}(H^-_{12}Q^+_{12}F_{12}+F^+_{12}Q^-_{12}H_{12}+Q^+_{12}F^-_{12}H_{12})=0\end{aligned}$$

(in order to show that Φ_{12} is a strong symmetry for $\hat{K}^0_{12}H_{12}$, where $\hat{K}^0_{12}=\hat{M}_{12}\doteq Q^-_{12}\sigma$, it is enough to replace H_{12} by σH_{12} in the previous calculation).

b) The Lie algebra of the starting operators (on H_{12}) is given by the following equations:

$$\begin{aligned}&[\hat{N}_{12}H^{(1)}_{12},\hat{N}_{12}H^{(2)}_{12}]_d=-\hat{N}_{12}H^{(3)}_{12},\qquad[\hat{N}_{12}H^{(1)}_{12},\hat{M}_{12}H^{(2)}_{12}]_d=-\hat{M}_{12}H^{(3)}_{12},\\&[\hat{M}_{12}H^{(1)}_{12},\hat{M}_{12}H^{(2)}_{12}]_d=-\hat{N}_{12}H^{(3)}_{12},\qquad H^{(3)}_{12}\doteq[H^{(1)}_{12},H^{(2)}_{12}]_l=(H^{(1)}_{12})^-H^{(2)}_{12},\end{aligned}\tag{C.16}$$

where

$$\hat{N}_{12}\doteq Q^-_{12},\qquad\hat{M}_{12}\doteq Q^-_{12}\sigma,\qquad P_{12}H^{(i)}_{12}=0,\quad H^{(i)}_{12}\ \text{diagonal},\quad i=1,2,3,$$

$$\begin{aligned}[Q^-_{12}H^{(1)}_{12},Q^-_{12}H^{(2)}_{12}]_d&=(Q^-_{12}H^{(2)}_{12})^-H^{(1)}_{12}-(Q^-_{12}H^{(1)}_{12})^-H^{(2)}_{12}\\&=-H^{(1)-}_{12}Q^-_{12}H^{(2)}_{12}+H^{(2)-}_{12}Q^-_{12}H^{(1)}_{12}\\&=-Q^-_{12}(H^{(1)}_{12})^-H^{(2)}_{12}.\end{aligned}$$

Equations (C.16b) and (C.16c) are obtained replacing $H^{(2)}_{12}$ by $\sigma H^{(2)}_{12}$ and $H^{(i)}_{12}$ by $\sigma H^{(i)}_{12}$, $i=1,2$, respectively, in the derivation of (C.16a).

c) The operator

$$\Phi_{12}\doteq\sigma(P_{12}-Q^+_{12}P^{-1}_{12}Q^+_{12}),\tag{C.17}$$

defined on off-diagonal matrices, is hereditary, namely

$$\Phi_{12_d}[\Phi_{12}F_{12}]G_{12}-\Phi_{12}\Phi_{12_d}[F_{12}]G_{12}\quad\text{is symmetric in } F_{12},G_{12}.\tag{C.18}$$

In order to show it, we make use of Eqs. (C.2), (C.15) and of

$$(\sigma F_{12})^{\pm}G_{12}=\begin{cases}\sigma F^{\pm}_{12}G_{12}, & G_{12}\ \text{diagonal},\\ \sigma F^{\mp}_{12}G_{12}, & G_{12}\ \text{off-diagonal}.\end{cases}\tag{C.19}$$

Here we display the calculations for the terms linear in $Q^{\pm}_{12}$:

$$\begin{aligned}
&-(\sigma P_{12}F_{12})^{+}P^{-1}_{12}Q^{+}_{12}G_{12}-Q^{+}_{12}P^{-1}_{12}(\sigma P_{12}F_{12})^{+}G_{12}\\
&\quad+\sigma P_{12}(F^{+}_{12}P^{-1}_{12}Q^{+}_{12}G_{12}+Q^{+}_{12}P^{-1}_{12}F^{+}_{12}G_{12})\\
&=\sigma(Q^{-}_{12}P^{-1}_{12}(P_{12}F_{12})^{-}G_{12}+F^{+}_{12}Q^{+}_{12}G_{12}+P_{12}Q^{+}_{12}P^{-1}_{12}F^{+}_{12}G_{12}),
\end{aligned}$$

which is symmetric in F_{12}, G_{12}, since

$$F^{+}_{12}G_{12}=G^{+}_{12}F_{12},$$

$$\begin{aligned}
&Q^{-}_{12}P^{-1}_{12}(P_{12}F_{12})^{-}G_{12}+F^{+}_{12}Q^{+}_{12}G_{12}\\
&\quad=Q^{-}_{12}F^{-}_{12}G_{12}+Q^{-}_{12}P^{-1}_{12}(P_{12}G_{12})^{-}F_{12}+F^{+}_{12}Q^{+}_{12}G_{12}\\
&\quad=G^{+}_{12}Q^{+}_{12}F_{12}+Q^{-}_{12}P^{-1}_{12}(P_{12}G_{12})^{-}F_{12}.
\end{aligned}$$

d) σ_{12} is an extended symmetry of

i) $Q_{1_t}=M^{(0)}_{11}=-2\sigma Q_1$, iff

$$\sigma_{12_f}[-2\sigma Q]=-2\sigma\delta_{12}, \tag{C.20a}$$

ii) $Q_{1_t}=M^{(1)}_{11}=-2Q_{1_x}$, iff

$$\sigma_{12_f}[-2Q_x]=-2D\sigma_{12}. \tag{C.20b}$$

$$\begin{aligned}
(\delta_{12}\hat{M}_{12}\cdot 1)_d[F_{12}]&=(Q^{-}_{12}\sigma\delta_{12})_d[F_{12}]=F^{-}_{12}\sigma\delta_{12}\\
&=-\sigma F^{+}_{12}\delta_{12}=-2\sigma F_{12}.
\end{aligned}$$

$$\begin{aligned}
&(\delta_{12}\hat{M}^{(1)}_{12})_d[F_{12}]=(\Phi_{12}Q^{-}_{12}\sigma\delta+2\alpha Q^{-}_{12}\sigma\delta'_{12})_d[F_{12}]\\
&\quad=\Phi_{12_d}[F_{12}]Q^{-}_{12}\sigma\delta_{12}+\Phi_{12}Q^{-}_{12_d}[F_{12}]\sigma\delta_{12}+2\alpha Q^{-}_{12_d}[F_{12}]\sigma\delta'_{12}\\
&\quad=-\sigma[(F^{+}_{12}P^{-1}_{12}Q^{+}_{12}+Q^{+}_{12}P^{-1}_{12}F^{+}_{12})Q^{-}_{12}\sigma\delta_{12}\\
&\qquad-(P_{12}-Q^{+}_{12}P^{-1}_{12}Q^{+}_{12})F^{-}_{12}\sigma\delta_{12}+F^{+}_{12}\delta'_{12}I]\\
&\quad=(-2P_{12}-2\alpha\sigma(D_1-D_2))F_{12}=-2DF_{12},
\end{aligned}$$

since, for instance,

$$\begin{aligned}
&\sigma P_{12}F^{-}_{12}\sigma\delta_{12}=-P_{12}F^{+}_{12}\delta_{12}I=-2P_{12}F_{12},\\
&-\sigma Q^{+}_{12}P^{-1}_{12}Q^{+}_{12}F^{-}_{12}\sigma\delta_{12}=Q^{-}_{12}P^{-1}_{12}Q^{-}_{12}F^{+}_{12}\delta_{12}I, \quad 2Q^{-}_{12}P^{-1}_{12}Q^{-}_{12}F_{12},\\
&F^{+}_{12}\delta'_{12}I=(D_1-D_2)F_{12},\\
&-\sigma Q^{+}_{12}P^{-1}_{12}F^{+}_{12}Q^{-}_{12}\sigma\delta_{12}=Q_{12}P^{-1}_{12}F_{12}Q^{+}_{12}\delta_{12}I=2Q_{12}P^{-1}_{12}F_{12}Q_{12}\\
&\quad=-2Q^{-}_{12}P^{-1}_{12}Q^{-}_{12}F_{12},
\end{aligned}$$

having used the properties

$$\begin{aligned}
&G^{\pm}_{12}\sigma=-\sigma G^{\mp}_{12}, \quad G_{12}\text{ off-diagonal},\\
&Q^{\pm}_{12}\sigma=-\sigma Q^{\mp}_{12},\\
&(I\delta^{n}_{12})^{\pm}F_{12}=(D^{n}_{1}\pm(-1)^{n}D^{n}_{2})F_{12}.
\end{aligned}$$

e) $\hat{\gamma}^0_{12}H_{12} \doteq \sigma\hat{K}^0_{12}H_{12}$ ($\hat{K}^0_{12}=\hat{N}_{12}$ and/or $\hat{M}_{12}$) are extended gradients, namely $(\hat{\gamma}^0_{12}H_{12})^*_d=(\hat{\gamma}^0_{12}H_{12})_d$.

i) If $\hat{\gamma}^0_{12}=\sigma\hat{N}_{12}=\sigma Q^-_{12}$, then $(\hat{\gamma}^0_{12}H_{12})_d[G_{12}]=\sigma G^-_{12}H_{12}=-\sigma H^-_{12}G_{12}$, and

$$\begin{aligned}\langle F_{12},(\hat{\gamma}^0_{12}H_{12})_d[G_{12}]\rangle &= -\langle F_{12},\sigma H^-_{12}G_{12}\rangle=\langle -\sigma H^-_{12}F_{12},G_{12}\rangle\\ &=\langle(\hat{\gamma}^0_{12}H_{12})_d[F_{12}],G_{12}\rangle;\end{aligned}$$

ii) If $\hat{\gamma}^0_{12}=\sigma\hat{M}_{12}=\sigma Q^-_{12}\delta=-Q^+_{12}$, then

$$(\hat{\gamma}^0_{12}H_{12})_d[G_{12}]=-G^+_{12}H_{12}=-H^+_{12}G_{12},$$

and

$$\begin{aligned}\langle F_{12},(\hat{\gamma}^0_{12}H_{12})_d[G_{12}]\rangle &= \langle F_{12},-H^+_{12}G_{12}\rangle=\langle -H^+_{12}F_{12},G_{12}\rangle\\ &=\langle(\hat{\gamma}^0_{12}H_{12})_d[F_{12}],G_{12}\rangle.\end{aligned}$$

Acknowledgements. It is a pleasure to acknowledge very useful discussions with M. J. Ablowitz and O. Ragnisco. One of the authors (P.M.S.) wishes to thank the Department of Mathematics and Computer Science of Clarkson University for its warm and friendly hospitality. This work was supported in part by the National Science Foundation under grant number DMS 8501325 and the Office of Naval Research under grant number N00014-76-C-0867.

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Communicated by A. Jaffe

Received December 30, 1986; in revised form August 10, 1987

Bi-Hamiltonian formulation of the Kadomtsev–Petviashvili and Benjamin–Ono equations

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(Received 11 June 1987; accepted for publication 21 October 1987)

It was shown recently that the Kadomtsev–Petviashvili (KP) equation (an integrable equation in 2 + 1, i.e., in two-spatial and one-temporal dimensions) admits a bi-Hamiltonian formulation. This was achieved by considering KP as a reduction of a (3 + 1)-dimensional system (in the variables x,y_1,y_2,t). It is shown here, using the KP as a concrete example, that equations in 2 + 1 possess *two* bi-Hamiltonian formulations and *two* recursion operators. Both Hamiltonian operators associated with the x direction are local; in contrast only one of the Hamiltonian operators associated with the y direction is local. Furthermore, using the Benjamin–Ono equation as a concrete example, it is shown that intergrodifferential equations in 1 + 1 admit an algebraic formulation analogous to that of equations in 2 + 1.

I. INTRODUCTION

This paper investigates symmetries, conserved quantities, recursion operators, mastersymmetries, and the bi-Hamiltonian formulation of two physically important exactly solvable evolution equations: the Kadomtsev–Petviashvili[1] (KP) and Benjamin–Ono[2,3] (BO) equations. The KP equation is a prototype integrable equation in 2 + 1 (i.e., in two-spatial and in one-temporal dimensions), while the BO equation is a prototype singular integrodifferential equation in 1 + 1. The results presented here fit in the general theory developed in Refs. 4 and 5; however, the following conceptual aspects are novel.

(i) Equations in *two* spatial dimensions (x and y) possess *two* recursion operators and *two* sets of compatible Hamiltonian operators. The set associated with the y direction was considered in Refs. 4–6. Here we investigate the recursion operator and the pair of *local* Hamiltonian operators associated with the x direction.

(ii) Integrodifferential equations in 1 + 1 share many common features with equations in 2 + 1.[7] This is because integrodifferential equations are also formulated in terms of two space operators, for example ∂_x and H (the Hilbert transform) in the case of the BO equation. It is shown here that the algebraic formulation of integrodifferential equations is analogous to that of equations in 2 + 1.

The existence of a double representation, corresponding to two recursion operators and two sets of bi-Hamiltonian operators, is also a property of integrodifferential equations in 1 + 1; this will be shown in a separate paper[8] for two explicit examples: the intermediate long wave[9,10] and the BO equations.

Hierarchies of infinitely many time-independent and time-dependent symmetries and conserved quantities of the KP equation have been obtained in Refs. 11 and 12. A recursion operator and a bi-Hamiltonian formulation of the KP were found in Refs. 4–6. This was achieved by introducing the following *extended* representation of the KP equation:

$$q_{1_t} = \int_{\mathbf{R}} dy_2\, \delta(y_1 - y_2) K_{12}, \quad q_1 = q(x,y_1,t), \tag{1.1}$$

where $\mathbf{R}$ denotes integration along the real axis, δ is the Dirac distribution, and K_{12} is some function of q_1 and $q_2 = q(x,y_2,t)$. The introduction of the above form is naturally motivated considering KP as a reduction of a nonlocal (3 + 1)-dimensional system (in the variables x, y_1, y_2, and t).[5,13] The above extension is necessary in order to bypass the Zakharov–Konopelchenko result on the nonexistence of recursion and bi-Hamiltonian operators in the usual (1 + 1)-dimensional formalism.[14]

Hierarchies of infinitely many time-independent and time-dependent symmetries and conserved quantities of the BO equation have been obtained in Refs. 12 and 15, via the mastersymmetry approach introduced by Fuchssteiner and one of the authors (A.S.F.). This approach was subsequently applied to the KP equation. It was shown in Ref. 5 that the mastersymmetry approach is contained in the general theory developed in Refs. 4 and 5.

A. Basic notions

We consider an evolution equation in its abstract form,

$$q_t = K(q), \tag{1.2}$$

on a normed space M of functions of $\mathbf{R}$; K is a suitable C^∞ vector field on M. We assume that the space of smooth vector fields on M is some space S of C^∞ functions on the real line or on the plane vanishing rapidly at infinity. By $K_f[v]$ we denote the Fréchet derivative of K in the direction v, i.e.,

$$K_f[v] \doteq \frac{\partial}{\partial \epsilon} K(q + \epsilon v)\Big|_{\epsilon=0}. \tag{1.3}$$

Let S^* be the dual of S with respect to the following bilinear form:

$$(\gamma,\sigma) \doteq \int_{\mathbf{R}} dx\, \gamma\sigma \quad \text{or} \quad (\gamma,\sigma) \doteq \int_{\mathbf{R}} dx\, dy\, \gamma\sigma, \tag{1.4}$$

$\gamma \in S^*$, $\sigma \in S$. Let $I: S \to \mathbf{R}$ be a functional, then its gradient is defined by

$$I_f[v] = (\text{grad}\, I, v). \tag{1.5}$$

It is well known that the function γ is a gradient of a func-

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 0022-2488/88/030604-14$02.50

tional I iff $\gamma_f = \gamma_f^+$, where the adjoint of an operator L is defined by $(L^+\gamma,\sigma) = (\gamma,L\sigma)$.

Definition 1.1: (i) A function $\sigma\in S$ is a *symmetry* of (1.2) iff the flow $q_t = \sigma$ commutes with the flow (1.2). This implies

$$\frac{\partial\sigma}{\partial t} + \sigma_f[K] - K_f[\sigma] = 0 . \tag{1.6}$$

(ii) A functional I is conserved by the flow (1.2) iff $dI/dt = 0$. Hence

$$\frac{\partial I}{\partial t} + (\gamma,K) = 0, \quad \gamma \doteq \text{grad}\, I ,$$

and $\gamma\in S^*$ is called a *conserved gradient* of (1.2). Differentiating the above equation in the arbitrary direction v it follows that γ satisfies

$$\frac{\partial\gamma}{\partial t} + \gamma_f[K] + K_f^+[\gamma] = 0, \quad \gamma_f = \gamma_f^+ . \tag{1.7}$$

(iii) Equation (1.2) is a *Hamiltonian system* iff it can be written in the form

$$q_t = \Theta f, \tag{1.8}$$

where f is a gradient function, i.e., $f_f = f_f^+$, and Θ is a Hamiltonian operator where

(1) Θ is skew symmetric, $\Theta^+ = -\Theta$,

(2) Θ satisfies a Jacobi identity, (1.9a)

$$(a,\Theta'[\Theta b]c) + \text{cyclic permutation} = 0 . \tag{1.9b}$$

A Hamiltonian operator Θ is associated with the Poisson bracket

$$\{I,H\} = (\text{grad}\, I, \Theta\, \text{grad}\, H) . \tag{1.9c}$$

(iv) An operator Φ is called a *recursion operator* or a *strong symmetry* of (1.2) iff it maps symmetries of (1.2) to symmetries of (1.2). Requiring that σ and $\Phi\sigma$ are symmetries of (1.2), it follows that an operator Φ satisfying the operator equation

$$\frac{\partial\Phi}{\partial t} + \Phi_f[K] + [\Phi,K_f] = 0 \tag{1.10}$$

is a recursion operator of (1.2).

(v) An operator Φ is called *hereditary* or *Nijenhuis* iff it generates an Abelian algebra. Assume that the flow $q_t = \sigma$ commutes with the flows $q_t = v$, $q_t = \Phi v$, and that the flow $q_t = v$ commutes with the flow $q_t = \Phi\sigma$, where σ, v are arbitrary functions. Requiring that the flows $q_t = \Phi\sigma$, $q_t = \Phi v$ also commute it follows that

$$\Phi_f[\Phi\sigma]v - \Phi\Phi_f[\Phi v]\sigma \text{ is symmetric w.r.t. } \sigma,v \tag{1.11}$$

(we have assumed for simplicity that $\partial\Phi/\partial t = 0$).

Exactly solvable evolution equations in 1 + 1 admit infinitely many symmetries. These symmetries are usually generated by a hereditary recursion operator Φ. An alternative approach is to use the notion of a *mastersymmetry*. A function τ is a master symmetry of Eq. (1.2) iff the map

$$[\tau,\cdot\,]_L, \quad \text{where } [\tau,\sigma]_L \doteq \tau_f[\sigma] - \sigma_f[\tau] ,$$

maps symmetries of (1.2). Here τ is called a *gradient mastersymmetry* (with respect to the invertible Hamiltonian operator Θ) iff $\Theta^{-1}\tau$ is a gradient function.

Integrable Hamiltonian systems in 1 + 1 have an exceptionally rich algebraic structure: They are *bi-Hamiltonian* systems. The existence of two Hamiltonian operators $\Theta^{(i)}$, $i = 1,2$, implies the existence of a *recursion operator* $\Phi \doteq \Theta^{(2)}(\Theta^{(1)})^{-1}$, which generates infinitely many symmetries, while Φ^+ generates infinitely many gradients of conserved quantities. For example, the two Hamiltonian operators associated with the Korteweg–de Vries (KdV) equation are given by

$$\Theta^{(1)} = D, \quad \Theta^{(2)} = D^3 + 2Dq + 2qD, \quad D \doteq \partial_x .$$

The KdV can be written as

$$q_t = q_{xxx} + 6qq_x = \Theta^{(1)}\gamma^{(1)} = \Theta^{(2)}\gamma^{(2)} ,$$

where

$$\gamma^{(2)} = q = \text{grad} \int_{\mathbb{R}} dx\, \frac{q^2}{2} ,$$

$$\gamma^{(1)} = q_{xx} + 3q^2 = \text{grad} \int_{\mathbb{R}} dx\, \frac{-q_x^2}{2 + q^3} .$$

Furthermore, $\Phi \doteq \Theta^{(2)}(\Theta^{(1)})^{-1}$ is a *recursion operator* for the KdV, i.e., Φ generates symmetries and Φ^+ generates gradients of conserved quantities. The KdV is the second member, $n = 1$, of the following Lax hierarchies:

$$q_t = \Phi^n q_x , \quad n = \text{non-negative integer} \tag{1.12}$$

(throughout this paper n,m,r denote non-negative integers), where q_x is a *starting symmetry*.

Exactly solvable equations in 2 + 1, written in the form (1.1), also admit a bi-Hamiltonian formulation.[4–6] For the KP, the two Hamiltonian operators are given by

$$\theta_{12}^{(1)} = D, \quad \theta_{12}^{(2)} = D^3 + Dq_{12}^+ + q_{12}^+ D + q_{12} D^{-1} q_{12} , \tag{1.13a}$$

where

$$\begin{aligned} &D \doteq \partial_x, \quad q_{12}^{\pm} \doteq q_1 \pm q_2 + \alpha(D_1 \mp D_2), \\ &D_i \doteq \partial_{y_i}, \quad i = 1,2, \end{aligned} \tag{1.13b}$$

and $q_i = q(x,y_i,t)$, $i = 1,2$. Indeed

$$\begin{aligned} q_{1_t} &= q_{1xxx} + 6q_1 q_{1_x} + 3\alpha^2 D^{-1} q_{1y_1y_1} \\ &\doteq K_{11} = \int_{\mathbb{R}} dy_2\, \delta_{12}\theta_{12}^{(1)}\gamma_{12}^{(1)} = \int_{\mathbb{R}} dy_2\, \delta_{12}\theta_{12}^{(2)}\gamma_{12}^{(2)} , \end{aligned} \tag{1.14}$$

where $\delta_{12} = \delta(y_1 - y_2)$ and $\gamma_{12}^{(i)}$, $i = 1,2$, are suitable extended gradients, i.e.,

$$I_d^{(i)}[v_{12}] = \langle\gamma_{12}^{(i)},v_{12}\rangle .$$

In the above the subscript d denotes a suitable directional derivative and $\langle\ ,\ \rangle$ denotes a suitable bilinear form.[4] Furthermore, the recursion operator $\phi_{12} \doteq \theta_{12}^{(2)}(\theta_{12}^{(1)})^{-1}$ generates *extended symmetries* σ_{12}, while the adjoint ϕ_{12}^* of ϕ_{12} with respect to $\langle\ ,\ \rangle$ generates *extended conserved gradients* γ_{12}. Then σ_{11}, γ_{11} are symmetries and conserved gradients of the KP, i.e., they satisfy Eqs. (1.6) and (1.7), respectively, where σ, γ, K are replaced by σ_{11}, γ_{11}, K_{11}, and K_{11} is defined in (1.14).

In analogy with Eq. (1.12), KP is the second member, $n=1$ ($\beta_1 = \frac{1}{2}$), of the following hierarchy:

$$q_{1_t} = \beta_n \int_{\mathbb{R}} dy_2\, \delta(y_1 - y_2) \phi_{12}^n \hat{M}_{12} \cdot 1 , \tag{1.15}$$

where $\hat{M}_{12} \cdot 1 = (Dq_{12}^+ + q_{12}^- D^{-1} q_{12}^+) \cdot 1$ is a *starting extended symmetry*. Actually the operator ϕ_{12} admits two starting symmetry operators $\hat{M}_{12}$ and $\hat{N}_{12} \doteq q_{12}$. They give rise to the following two hierarchies of *time-independent* symmetries:

$$(\phi_{12}^n \hat{M}_{12} \cdot 1)_{11}, \quad (\phi_{12}^m \hat{N}_{12} \cdot 1)_{11} . \tag{1.16}$$

Time-dependent symmetries of order r of the KP are produced by linear combinations of

$$(\phi_{12}^n \hat{M}_{12} \cdot (y_1 + y_2)^r)_{11}, \quad (\phi_{12}^m \hat{N}_{12} \cdot (y_1 + y_2)^r)_{11} , \tag{1.17}$$

and are closely related to gradient mastersymmetries. The above hierarchies of time-independent and time-dependent symmetries give rise to time-independent and time-dependent conserved quantities.[4–6] Finally, there exists a simple relationship between ϕ_{12} and a *nongradient mastersymmetry* T_{12}:

$$T_{12} = \phi_{12}^2 \cdot \delta(y_1 - y_2), \quad C\phi_{12} = T_{12_d} + DT_{12_d}^* D^{-1} , \tag{1.18}$$

where C is a constant. The above equations are the two-dimensional analogs of the following formulas, valid for the KdV:

$$T = \phi \cdot 1, \quad C\phi = T_f + DT_f^* D^{-1} . \tag{1.19}$$

It is well known that the KP equation is associated with the linear problem

$$w_{xx} + (q(x,y,t) + \alpha\, \partial_y) w = 0 . \tag{1.20}$$

The recursion operator ϕ_{12} is algorithmically derived from Eq. (1.20).[4,6]

B. New results

(i) The KP equation: In Refs. 4–6 the algebraic properties of KP were investigated by expanding in terms of $\delta(y_1 - y_2)$. Now we expand in terms of $\delta(x_1 - x_2)$ (Ref. 16) and write KP in the form

$$q_{1_t} = \int_{\mathbb{R}} dx_2\, \delta(x_1 - x_2) K_{12}, \quad q_1 = q(x_1, y, t) , \tag{1.21}$$

where K_{12} is some function of q_1, $q_2 = q(x_2, y, t)$. Let subscripts 12 denote dependence on x_1, x_2, y; then for arbitrary functions f_{12}, g_{12} we define the following bilinear form:

$$\langle f_{12}, g_{12} \rangle \doteq \int_{\mathbb{R}^3} dx_1\, dx_2\, dy\, f_{21} g_{12} . \tag{1.22}$$

Let the arbitrary operator $\hat{K}_{12}$ depend on the operators q_{12}^+, q_{12}^-, where

$$q_{12}^{\pm} \doteq q_1 \pm q_2 + D_1^2 \pm D_2^2, \quad D_i = \partial_{x_i} , \qquad q_i = q(x_i, y, t), \quad i = 1,2 ; \tag{1.23}$$

then the directional derivative of $\hat{K}_{12}$ in the direction σ_{12} is denoted by $\hat{K}_{12_d}[\sigma_{12}]$ and is defined by

$$\hat{K}_{12_d}[\sigma_{12}] f_{12} \doteq \frac{\partial \hat{K}_{12}}{\partial \epsilon}(q_{12}^+ + \epsilon \sigma_{12}^+, q_{12}^- + \epsilon \sigma_{12}^-) f_{12} \Big|_{\epsilon = 0} , \tag{1.24a}$$

where

$$\sigma_{12}^{\pm} f_{12} \doteq \int_{\mathbb{R}} dx_3 (\sigma_{13} f_{32} \pm \sigma_{32} f_{13}) . \tag{1.24b}$$

The two Hamiltonian operators associated with the KP equation (1.21) are given by

$$\Theta_{12}^{(1)} \doteq D_1 + D_2, \quad \Theta_{12}^{(2)} = \alpha\, \partial_y + q_{12}^- , \tag{1.25}$$

where $q_{12}^{\pm}$ are defined in (1.23). The operators $\Theta_{12}^{(i)}$, $i = 1,2$, are skew symmetric, and satisfy the Jacobi identity

$$\langle a_{12}, \Theta_{12_d}^{(i)}[b_{12}] c_{12} \rangle + \text{cyclic permutation} = 0 , \tag{1.26}$$

where Θ_{12_d} and $\langle\ ,\ \rangle$ are defined by (1.22)–(1.24).

It should be stressed that, in contrast to the Hamiltonian pair (1.13), both of the above Hamiltonian operators are *local.* The KP is a bi-Hamiltonian system,

$$q_{1_t} = q_{1x_1x_1x_1} + 6q_1 q_{1_{x_1}} + 3\alpha^2 D_1^{-1} q_{1_{yy}} = K_{11} = \int_{\mathbb{R}} dx_2\, \delta_{12} \Theta_{12}^{(i)} \gamma_{12}^{(i)}, \quad i = 1,2 , \tag{1.27}$$

where $\gamma_{12}^{(i)}$ are appropriate extended gradients.

KP is the fourth member, $n = 3$, of the following Lax hierarchy:

$$q_{1_t} = \beta_n \int_{\mathbb{R}} dx_2\, \delta(x_1 - x_2) \Phi_{12}^n \hat{K}_{12}^0 \cdot 1 , \tag{1.28}$$

where

$$\Phi_{12} \doteq \Theta_{12}^{(2)} (\Theta_{12}^{(1)})^{-1}, \quad \hat{K}_{12}^0 \doteq \alpha\, \partial_y + q_{12}^- . \tag{1.29}$$

The recursion operation Φ_{12} admits only one starting symmetry operator $\hat{K}_{12}^0$, which generates the time-independent symmetries $(\Phi_{12}^m \hat{K}_{12}^0 \cdot 1)_{11}$. Values of m zero or even correspond to (1.16a), while m odd corresponds to (1.16b). Thus in the new formulation the two different hierarchies obtained in Ref. 4 are unified. Similarly Φ_{12}^* generates extended conserved gradients $\gamma_{12}^{(m)}$, which give rise to conserved gradients $\gamma_{11}^{(m)}$.

A nongradient mastersymmetry is given by

$$\Phi_{12}^2 \cdot \frac{\partial \delta(x_1 - x_2)}{\partial x_1} .$$

The recursion operator Φ_{12} can also be algorithmically obtained from the linear equation (1.20).

(ii) The BO equation: The BO equation

$$q_t = 2qq_x + Hq_{xx}, \quad q = q(x,t) , \tag{1.30a}$$

where H denotes the Hilbert transform (throughout this paper principal value integrals are assumed if needed)

$$(Hf)(x) \doteq \pi^{-1} \int_{\mathbb{R}} d\xi (\xi - x)^{-1} f(\xi) , \tag{1.30b}$$

can be written in the form

$$q_{1_t} = \int_{\mathbb{R}} dx_2\, \delta(x_1 - x_2) K_{12}, \quad q_1 = q(x_1, t) , \tag{1.31}$$

where K_{12} is some function of q_1, $q_2 = q(x_2, t)$. Let subscript

12 denote dependence on x_1, x_2; then for arbitrary functions f_{12}, g_{12} we define the following bilinear form:

$$\langle f_{12}, g_{12}\rangle \doteq \int_{\mathbb{R}^2} dx_1\, dx_2\, f_{21} g_{12}\,. \tag{1.32}$$

Let the arbitrary operator $\hat{K}_{12}$ depend on the operators q_{12}^{+}, q_{12}^{-}, where

$$q_{12}^{\pm} \doteq q_1 \pm q_2 + i(D_1 \mp D_2), \quad D_i \doteq \partial_{x_i}\,, \qquad q_i = q(x_i,t), \quad i = 1,2\,; \tag{1.33}$$

then the directional derivative of $\hat{K}_{12}$ in the arbitrary direction σ_{12} is denoted by $\hat{K}_{12_d}[\sigma_{12}]$ and is defined by (1.24).

Two compatible Hamiltonian operators associated with the BO equation are given by

$$\Theta_{12}^{(1)} \doteq q_{12}^{-}, \quad \Theta_{12}^{(2)} = (q_{12}^{+} - i q_{12}^{-} H_{12}) q_{12}^{-}\,, \tag{1.34a}$$

where the operator H_{12} is an extended H operator,

$$(H_{12} f)(x_1,x_2) \doteq \pi^{-1} \int_{\mathbb{R}} d\xi\, [\xi - (x_1 + x_2)]^{-1} \times F(\xi, x_1 - x_2)\,, \tag{1.34b}$$

and $f(x_1,x_2) = F(x_1 + x_2, x_1 - x_2)$. The BO equation is a bi-Hamiltonian system with respect to the above Hamiltonian operators.

The BO equation is a member of the following Lax hierarchy:

$$q_{1_t} = \beta_n \int_{\mathbb{R}} dx_2\, \delta(x_1 - x_2) \Phi_{12}^n q_{12}^{-} \cdot 1, \qquad \Phi_{12} \doteq q_{12}^{+} - i q_{12}^{-} H_{12}\,. \tag{1.35}$$

Indeed, (1.35) with $n = 1$ and $n = 2$ yields

$$q_{1_t} = 2i\beta_0 q_{1_{x_1}}, \quad q_{1_t} = 4i\beta_1 (2 q_1 q_{1_{x_1}} + H_1 q_{1 x_1 x_1})\,. \tag{1.36}$$

The operator $\Phi_{12} = \Theta_{12}^{(2)} (\Theta_{12}^{(1)})^{-1}$ generates the time-independent symmetries of the BO equation $(\Phi_{12}^m q_{12}^{-} \cdot 1)_{11}$. Similarly, Φ_{12}^{*} generates extended conserved gradients $\gamma_{12}^{(m)}$.

The above recursion operator Φ_{12} can be derived algorithmically from the associated linear problem of the BO equation.

This paper is organized as follows. In Sec. II we derive the *second representation* of the KP class and we investigate the algebraic properties of the associated recursion operator and bi-Hamiltonian operators. In Sec. III we derive the extended representation of the BO class and we investigate the algebraic properties of the associated recursion and bi-Hamiltonian operators. In addition we discuss the connection with the mastersymmetries theory of the BO equation and with the complex Burgers hierarchy.

II. THE KP EQUATION

A. Derivation of the second representation

Proposition 2.1: The linear equation

$$-\alpha w_y = \hat{q} w, \quad \hat{q} \doteq q(x,y,t) + \partial_x^2\,, \tag{2.1}$$

is associated with the Lax hierarchy

$$q_{1_t} = \beta_n \int_{\mathbb{R}} dx_2\, \delta(x_1 - x_2) \Phi_{12}^n \hat{K}_{12}^0 \cdot 1$$
$$= \beta_n \int_{\mathbb{R}} dx_2\, \delta(x_1 - x_2)(D_1 + D_2) \Psi_{12}^{n+1} \cdot 1\,, \tag{2.2}$$

where β_n are constants, $D_i \doteq \partial_{x_i}$, $i = 1,2$, and the operators Φ_{12}, Ψ_{12}, $\hat{K}_{12}^0$ are defined by

$$\Phi_{12} \doteq (\alpha\, \partial_y + q_{12}^{-})(D_1 + D_2)^{-1}\,, \qquad (D_1 + D_2)\Psi_{12} = \Phi_{12}(D_1 + D_2)\,, \tag{2.3a}$$

$$q_{12}^{\pm} = \hat{q}_1 \pm \hat{q}_2, \quad \hat{K}_{12}^0 = \alpha\, \partial_y + q_{12}^{-}\,. \tag{2.3b}$$

Remark 2.1: (i) $\hat{q}_2 = \hat{q}_1^{*}$, where * denotes the adjoint with respect to the bilinear form (1.22).

(ii) $\Psi_{12} = \Phi_{12}^{*}$.

(iii) Equation (2.2) with $n = 0,1,2,3$ and $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{1}{4}$, $\beta_3 = \frac{1}{2}$ implies

$$q_{1_t} = 0, \quad q_{1_t} = q_{1_{x_1}}, \quad q_{1_t} = \alpha q_{1_y}, \qquad q_{1_t} = q_{1_{x_1 x_1 x_1}} + 6 q_1 q_{1_{x_1}} + 3\alpha^2 D_1^{-1} q_{1_{yy}}\,. \tag{2.4}$$

Thus both the x-translation and the y-translation hierarchies of the KP are generated by the same extended starting symmetry $\hat{K}_{12}^0 \cdot 1 = q_1 - q_2$.

To derive the above Lax hierarchy we look for compatible flows

$$w_t = V w, \quad V \text{ polynomial in } \partial_x\,. \tag{2.5}$$

Compatibility of (2.1), (2.5) implies the *operator* equation

$$q_t = -(\alpha V_y + [q + \partial_x^2, V])\,. \tag{2.6}$$

Assuming the integral representation

$$(Vf)(x_1,y) = \int_R dx_2\, v(x_1,x_2,y) f(x_2,y), \quad v_{12} \doteq v(x_1,x_2,y) \tag{2.7}$$

and noting that

$$(q_1 + D_1^2) V_1 f_1 = \int_{\mathbb{R}} dx_2 \{(q_1 + D_1^2) v_{12}\} f_2\,,$$

$$V_1 (q_1 + D_1^2) f_1 = \int_{\mathbb{R}} dx_2 \{(q_2 + D_2^2) v_{12}\} f_2\,,$$

$$V_y f = \int_{\mathbb{R}} dx_2\, v_{12y} f_2\,,$$

we obtain the distribution equation

$$q_{1_t} \delta_{12} = -(\alpha v_{12_y} + q_{12}^{-} v_{12})\,. \tag{2.8}$$

Thus

$$q_{1_t} \delta_{12} = -(D_1 + D_2) \Psi_{12} v_{12}\,, \qquad \Psi_{12} \doteq (D_1 + D_2)^{-1} (\alpha\, \partial_y + q_{12}^{-})\,. \tag{2.9}$$

The operator $(D_1 + D_2)\Psi_{12}$ satisfies the following commutator operator equation:

$$[(D_1 + D_2)\Psi_{12}, h_{12}] = 2 h_{12}' (D_1 + D_2)\,, \qquad h_{12} = h(x_1 - x_2), \quad h_{12}' = \frac{d}{dx_1} h_{12}\,. \tag{2.10}$$

Using the above equation and assuming the expansion

$$v_{12} = \sum_{j=0}^{n} \delta^j_{12} v^{(j)}_{12}, \quad \delta^j_{12} = \frac{d^j}{dx^j}\delta_{12}, \tag{2.11}$$

Eq. (2.9) yields

$$q_{1_s}\delta_{12} = \sum_{j=0}^{n} \delta^j_{12}(D_1 + D_2)\Psi_{12}v^{(j)}_{12}$$
$$+ 2\sum_{j=1}^{n+1} \delta^j_{12}(D_1 + D_2)v^{(j-1)}_{12}.$$

Thus

$$(D_1 + D_2)v^{(n)}_{12} = 0, \quad q_{1_s}\delta_{12} = \delta_{12}(D_1 + D_2)\Psi_{12}\, v^{(0)}_{12},$$
$$-\tfrac{1}{2}\Psi_{12}v^{(j)}_{12} = v^{(j-1)}_{12}.$$

Therefore $v^{(0)}_{12} = (-\tfrac{1}{2})^n \Psi^n_{12} v^{(n)}_{12}$. Hence assuming $v^{(n)}_{12} = 1$, the above equations imply

$$q_{1_s}\delta_{12} = \delta_{12}(D_1 + D_2)\,\Psi^{n+1}_{12}\cdot 1$$
$$= \delta_{12}\Phi^n_{12}\Phi_{12}(D_1 + D_2)\cdot 1,$$

where

$$(D_1 + D_2)\Psi_{12} = \Phi_{12}(D_1 + D_2).$$

B. Isospectrality yields a hereditary operator

To make this paper self-contained we first introduce an appropriate directional derivative. Recall the integral representation [Eq. (2.7)],

$$(Vf)(x_1,y) = \int_{\mathbb{R}} dx_3\, v(x_1,x_3,y)f(x_3,y).$$

Also, allowing f to depend on x_2 we obtain $Vf_{12} = \int_{\mathbb{R}} dx_3\, v_{13}f_{32}$. In particular,

$$\hat{q}_1 f_{12} = (q_1 + D_1^2)f_{12} = \int_{\mathbb{R}} dx_3\, q_{13}\, f_{32}. \tag{2.12}$$

Equation (2.12) is a map between an operator and its kernel and induces the following directional derivative:

$$\hat{q}_{1_d}[\sigma_{12}]f_{12} = \int_{\mathbb{R}} dx_3\, \sigma_{13}f_{32}. \tag{2.13}$$

Equation (2.12) and the bilinear form (1.22) imply that the adjoint of $\hat{q}_1$, $\hat{q}^*_1 = q_2 + D_2^2$ has the representation

$$\hat{q}^*_1 f_{12} = (q_2 + D_2^2)f_{12} = \int_{\mathbb{R}} dx_3\, q_{32}f_{13}. \tag{2.14}$$

Hence

$$\hat{q}^*_{1_d}[\sigma_{12}]f_{12} = \int_{\mathbb{R}} dx_3\, \sigma_{32}\, f_{13}. \tag{2.15}$$

Equations (2.12)–(2.15) and $q^\pm_{12} \doteq \hat{q}_1 \pm \hat{q}^*_1$ imply (1.24).

Proposition 2.2: (i) Consider the isospectral equation

$$\hat{q}v + \alpha v_y = \lambda v, \tag{2.16a}$$

and its adjoint, with respect to the bilinear form (1.4),

$$\hat{q}v^+ - \alpha v^+_y = \lambda v^+. \tag{2.16b}$$

Then

$$(\text{grad}\,\lambda)_{12} = v_1 v_2^+, \tag{2.17}$$

where $(\text{grad}\,\lambda)_{12}$ denotes the gradient of λ with respect to the bilinear form (1.22).

(ii) Equations (2.16) imply

$$(\alpha\,\partial_y + q_{12})v_1 v_2^+ = 0. \tag{2.18}$$

To derive the above results, take the directional derivative of (2.16a) in the arbitrary direction f_{12}, multiply this equation by v_1^+ and integrate over $dx\, dy$ to obtain

$$\lambda_d[f_{12}] = \int_{\mathbb{R}^2} dx_1\, dy\, v_1^+\, \hat{q}_{1_d}[f_{12}]v_1.$$

Using (2.13) the above becomes

$$\lambda_d[f_{12}] = \int_{\mathbb{R}^3} dx_1\, dx_2\, dy\, v_1^+ v_2 f_{12}.$$

But

$$\lambda_d[f_{12}] = \int_{\mathbb{R}^3} dx_1\, dx_2\, dy (\text{grad}\,\lambda)_{21} f_{12},$$

hence (2.17) follows. Equation (2.18) is a trivial consequence of (2.16).

Equation (2.18) suggests that Φ_{12} is a hereditary (Nijenhuis) operator (see Proposition 4.3 of Ref. 4). Actually it can be easily verified that

$$\Phi_{12_d}[\Phi_{12}f_{12}]g_{12} - \Phi_{12}\Phi_{12_d}[f_{12}]g_{12}$$
$$\text{is symmetric w.r.t. } f_{12}, g_{12}, \tag{2.19}$$

i.e., Φ_{12} is indeed hereditary (see Appendix A).

C. Symmetries and conserved gradients

1. Starting symmetries

We recall that the starting symmetry operators play an important role in the theory developed in Refs. 4 and 5. An operator Φ_{12} algorithmically implies starting symmetry operators: Look for operators $\hat{S}_{12}$ such that $\hat{S}_{12}H_{12} = 0$, but $\Phi_{12}\hat{S}_{12}H_{12} \neq 0$. Then a starting symmetry operator $\hat{K}^0_{12}$ is given by $\hat{K}^0_{12}H_{12} \doteq \Phi_{12}\hat{S}_{12}H_{12}$.

Proposition 2.3: Let

$$\hat{K}^0_{12} \doteq \alpha\,\partial_y + q_{12}, \quad H_{12} \doteq H(x_1 - x_2, y), \tag{2.20}$$

where H is an arbitrary function of the arguments indicated. Then the following statements obtain.

(i) $\hat{K}^0_{12}\cdot H_{12}$ is a starting symmetry associated with the operator Φ_{12} [defined in (2.3)].

(ii) $\hat{K}^0_{12}$ satisfies a simple commutator operator equation with $h_{12} \doteq h(x_1 - x_2)$,

$$[\hat{K}^0_{12}, h_{12}] = 2\frac{\partial h_{12}}{\partial_{x_1}}(D_1 + D_2). \tag{2.21}$$

(iii) Φ_{12} is a strong symmetry for $\hat{K}^0_{12}\cdot H_{12}$, i.e.,

$$\mathcal{L}(\Phi_{12}, \hat{K}^0_{12}H_{12})$$
$$\doteq \Phi_{12d}[\hat{K}^0_{12}H_{12}] + [\Phi_{12}, (\hat{K}^0_{12}H_{12})_d] = 0. \tag{2.22}$$

(iv) The Lie algebra of the starting symmetry operator satisfies

$$[\hat{K}^0_{12}H^{(1)}_{12}, \hat{K}^0_{12}H^{(2)}_{12}]_d = \hat{K}^0_{12}[H^{(1)}_{12}, H^{(2)}_{12}]_I, \tag{2.23}$$

where

$$[K^{(1)}_{12}, K^{(2)}_{12}]_d \doteq K^{(1)}_{12_d}[K^{(2)}_{12}] - K^{(2)}_{12_d}[K_{12}], \tag{2.24a}$$

$$[H^{(1)}_{12}, H^{(2)}_{12}]_I \doteq \int_{\mathbb{R}} dx_3 (H^{(1)}_{13}H^{(2)}_{32} - H^{(2)}_{13}H^{(1)}_{32}). \tag{2.24b}$$

To derive (i) let $S_{12} = D_1 + D_2$; then H_{12} is defined by $(D_1 + D_2)H_{12} = 0$, thus $H_{12} = H(x_1 - x_2, y)$. Also $\hat{K}^0_{12} H_{12} = (\alpha \partial_y + q_{12}^-)H_{12}$. Part (ii) is a straightforward calculation and part (iii) follows from the definition of a starting symmetry and the fact that Φ_{12} is hereditary (see Lemma 4.2 of Ref. 4). Part (iv) is a tedious calculation [see Appendix A for a direct verification of Eq. (2.22) and (2.23)].

2. Symmetries

We recall that σ_{12} is a time-independent extended symmetry of Eq. (2.2) iff

$$[\delta_{12}\Phi^n_{12}\hat{K}^0_{12} \cdot 1, \sigma_{12}]_d = 0 . \tag{2.25}$$

Proposition 2.4:

(i) $$\delta_{12}\Phi^n_{12}\hat{K}^0_{12} \cdot 1 = \sum_{l=0}^{n} b_{n,l}\Phi^{n-l}_{12}\hat{K}^0_{12}\delta^l_{12} ,$$

$b_{n,l}$ constants . (2.26)

(ii) $$[\delta_{12}\Phi^n_{12}\hat{K}^0_{12} \cdot 1, \Phi^m_{12}\hat{K}^0_{12} \cdot H_{12}]_d = \sum_{l=0}^{n} b_{n,l}\Phi^{n-l+m}_{12}\hat{K}^0_{12}[\delta^l_{12}, H_{12}]_l . \tag{2.27}$$

(iii) $\sigma^{(m)}_{12} \doteq \Phi^m_{12}\hat{K}^0_{12} \cdot 1$ are time-independent extended symmetries of (2.2) .

(iv) $\sigma^{(m)}_{11}$ are symmetries of (2.2) .

(v) $\sigma^{(m)}_{12} = 0$ are auto-Bäcklund transformations of (2.2), where q_1, q_2 are interpreted to be two different solutions of (2.2).

Part (i) of the above follows from

$$[\Phi_{12}, h_{12}] = 2h'_{12}, \quad [\hat{K}^0_{12}, h_{12}] = 2h'_{12}(D_1 + D_2) . \tag{2.28}$$

To derive (ii) note that

$$\begin{aligned}
&[\delta_{12}\Phi^n_{12}\hat{K}^0_{12} \cdot 1, \Phi^m_{12}\hat{K}^0_{12} \cdot H_{12}]_d \\
&= \sum_{l=0}^{n} b_{n,l}[\Phi^{n-l}_{12}\hat{K}^0_{12}\delta^l_{12}, \Phi^m_{12}\hat{K}^0_{12}H_{12}]_d \\
&= \sum_{l=0}^{n} b_{n,l}\Phi^{n-l+m}_{12}[\hat{K}^0_{12}\delta^l_{12}, \hat{K}^0_{12}H_{12}]_d \\
&= \sum_{l=0}^{n} b_{n,l}\Phi^{n-l+m}_{12}\hat{K}^0_{12}[\delta^l_{12}, H_{12}]_l ,
\end{aligned}$$

where we have used (for the third equality) the fact that Φ is hereditary and a strong symmetry for $\hat{K}^0_{12} \cdot H_{12}$, and the fourth equality follows from Eq. (2.23). Part (iii) follows from (ii) by taking $H_{12} = 1$. Part (iv) follows from (iii) and (2.8) (see Theorem 4.1 of Ref. 4). For part (v) see Theorem 4.2 of Ref. 4.

Remark 2.1: (i) Using Eq. (2.27) with suitable functions H_{12}, it should be possible to show that time-dependent symmetries of (2.2) are generated by linear combinations of $\Phi^m_{12}\hat{K}^0_{12}H_{12}$. See Ref. 5 for the corresponding results associated with the first representation.

(ii) An analysis about conserved gradients should follow closely the methods developed in Refs. 4 and 5. For example, it can be shown that $\Psi^n_{12} \cdot H_{12}$ are extended gradients for all $H_{12} = H(x_1 - x_2, y)$.

3. A nongradient mastersymmetry

Proposition 2.5: (i) $T_{12} \doteq \Phi^2_{12}\delta^1_{12}$, $\delta^1_{12} \doteq \partial\delta(x_1 - x_2)/\partial x_1$ is a nongradient mastersymmetry of the KP class, since

$$[\Phi^n_{12}\hat{K}^0_{12}H_{12}, T_{12}]_d = (n+1)\Phi^{n+1}_{12}\hat{K}^0_{12}H_{12} . \tag{2.29}$$

(ii) T_{12} generates the recursion operator Φ_{12} via

$$2\Phi_{12} = T_{12_d} + (D_1 + D_2)T^*_{12_d}(D_1 + D_2)^{-1} . \tag{2.30}$$

(iii) Let

$$\hat{\gamma}^{(n)}_{12} \doteq (\Phi^*_{12})^n\hat{\gamma}^0_{12}, \quad \hat{\gamma}^0_{12} \doteq (D_1 + D_2)^{-1}\hat{K}^0_{12} . \tag{2.31}$$

Then

$$\hat{\gamma}^{(n)}_{12}H_{12} = \mathrm{grad}_{12}\, I_n , \tag{2.32a}$$

$$I \doteq 1/(n+2)\langle \hat{\gamma}^{(n+1)}_{12}H_{12}, \delta^1_{12}\rangle . \tag{2.32b}$$

The proof of (i)–(iii) is a consequence of equations $\delta^1_{12_d} = 0$, $\Phi_{12_d}[\delta^1_{12}] = 1$ and of Eq. (4.9), (4.6), and (4.7) of Ref. 5, respectively.

III. THE BO EQUATION

The linear problem associated with the BO equation (1.30) is the following differential Riemann–Hilbert (RH) boundary value problem:

$$\psi^{(-)}(x) = (q(x) + i\partial_x)\psi^{(+)} , \tag{3.1}$$

where $\psi^{(+)}$ and $\psi^{(-)}$ are the boundary values on the line $\mathrm{Im}\, x = 0$ of functions holomorphic in the upper and lower half-plane, respectively,[17] and the spectral parameter has been rescaled away.

Equation (3.1) plays a crucial role in the derivation of the recursion and bi-Hamiltonian operators of the BO class.

A. Derivation of the recursion and bi-Hamiltonian operators

Proposition 3.1: The linear problem (3.1) is associated with the hierarchy

$$\begin{aligned}
q_{1_{t_n}} &= \beta_n \int_{\mathbb{R}} dx_2\, \delta(x_1 - x_2)\, \Phi^n_{12}\hat{K}^0_{12} \cdot 1 \\
&= \beta_n \int_{\mathbb{R}} dx_2\, \delta(x_1 - x_2)\, q_{12}^-\Psi^n_{12} \cdot 1 ,
\end{aligned} \tag{3.2}$$

where β_n are constants and the operators Φ_{12}, Ψ_{12}, and $\hat{K}^0_{12}$ are defined by

$$\Phi_{12} \doteq q^+_{12} - iq^-_{12}H_{12}, \quad q^-_{12}\Psi_{12} = \Phi_{12}q^-_{12}, \quad \hat{K}^0_{12} \doteq q^-_{12} , \tag{3.3a}$$

$$H_{12}f_{12} \doteq \pi^{-1}\int_{\mathbb{R}} d\xi\, [\xi - (x_1 + x_2)]^{-1}F(\xi, x_1 - x_2) ,$$

$$f_{12} \doteq f(x_1, x_2) = F(x_1 + x_2, x_1 - x_2) , \tag{3.3b}$$

$$\begin{aligned}
&q^{\pm}_{12} \doteq q_1 \pm q_2 + i(D_1 \mp D_2), \quad q_i = q(x_i, t) , \\
&D_i = \partial_{x_i}, \quad i = 1,2 .
\end{aligned} \tag{3.3c}$$

Remark 3.1: (i) $\Psi_{12} = \Phi^*_{12}$, where * denotes the adjoint with respect to the bilinear form (1.32).

(ii) The first few equations of the BO hierarchy are then

$$q_t = 0, \quad n = 0, \tag{3.4a}$$

$$q_t = q_x, \quad n = 1, \quad \beta_1 = (2i)^{-1} \quad \text{(wave equation)}, \tag{3.4b}$$

$$q_t = 2qq_x + Hq_{xx}, \; n = 2, \; \beta_2 = (4i)^{-1} \quad \text{(BO equation)}, \tag{3.4c}$$

$$q_t = (-q_{xx} + q^3 + \tfrac{3}{2}(qHq_x + Hqq_x))_x,$$
$$n = 3, \quad \beta_3 = (8i)^{-1} \quad \text{(higher-order BO equation)}, \tag{3.4d}$$

and are obtained from (3.2) using Eqs. (3.16b)–(3.16f).

To derive the representation (3.2) we first seek compatibility between the differential RH problem (3.1) and the evolution equations

$$\psi_t^{(\pm)} = V^{(\pm)}\psi^{(\pm)}, \tag{3.5$_\pm$}$$

where $V^{(\pm)}$ are differential operators of the form

$$V^{(\pm)} = \sum_{j=0}^{n} V_j^{(\pm)}(x)\partial_x^j \tag{3.6}$$

and the coefficients $V_j^{(+)}(x)$ and $V_j^{(-)}(x)$ are holomorphic in the upper and lower half x plane, respectively.

The compatibility condition between (3.1) and (3.5) yields the operator equation

$$q_t = V^{(-)}(q + i\,\partial_x) - (q + i\,\partial_x)V^{(+)}, \tag{3.7}$$

which can be converted into a scalar distribution equation by formally introducing the integral representation

$$(V^{(\pm)}f)(x_1) = \int_{\mathbf{R}} dx_2\, v_{12}^{(\pm)}f(x_2), \quad v_{12}^{(\pm)} \doteq v^{(\pm)}(x_1,x_2). \tag{3.8}$$

For instance, the operator $V_1^{(-)}(q_1 + i\,\partial_{x_1})$ gives rise to the scalar kernel $(q_2 - i\,\partial_{x_2})v_{12}^{(-)}$, since

$$V_1^{(-)}(q_1 + i\,\partial_{x_1})f(x_1) = \int_{\mathbf{R}} dx_2\, v_{12}^{(-)}(q_2 + i\,\partial_{x_2})f(x_2)$$
$$= \int_{\mathbf{R}} dx_2[(q_2 - i\,\partial_{x_2})v_{12}^{(-)}]f(x_2). \tag{3.9}$$

Equation (3.7) then corresponds to the following distribution scalar equation:

$$\delta(x_1 - x_2)\,q_{1_t} = -(q_1 + i\,\partial_{x_1})v_{12}^{(+)} + (q_2 - i\,\partial_{x_2})v_{12}^{(-)}$$
$$= -\tfrac{1}{2}(q_{12}^{+}(v_{12}^{(+)} - v_{12}^{(-)})$$
$$+ q_{12}^{-}(v_{12}^{+} + v_{12}^{(-)})). \tag{3.10}$$

Equations (3.6), (3.8), and (3.10) imply for $v_{12}^{(\pm)}$ the following expansions in derivatives of δ_{12}:

$$v_{12}^{(\pm)} = \sum_{j=0}^{n} \delta_{12}^{j} v_{12}^{(\pm)j}. \tag{3.11}$$

Combining (3.11), (3.8), and the analyticity properties of $V_j^{(\pm)}(x)$, we obtain that $v_{12}^{(+)}$ and $v_{12}^{(-)}$ are holomorphic in the upper and lower $x_1 + x_2$ plane, respectively. Then, in particular,

$$v_{12}^{(+)} - v_{12}^{(-)} = -iH_{12}(v_{12}^{(+)} - v_{12}^{(-)}) \tag{3.12}$$

[see Eq. (3.19)], and Eq. (3.10) becomes

$$\delta_{12}q_{1_t} = -\tfrac{1}{2}\Phi_{12}\tilde{v}_{12}, \quad \tilde{v}_{12} \doteq v_{12}^{(+)} - v_{12}^{(-)}. \tag{3.13}$$

Remark 3.2: The following operator commutator equations hold:

$$[q_{12},h_{12}] = [H_{12},h_{12}] = 0,$$
$$[q_{12}^{+},h_{12}] = [\Phi_{12},h_{12}] = 2ih_{12}', \quad h_{12}' \doteq \frac{\partial h_{12}}{\partial x_1}, \tag{3.14}$$

and hereafter h_{12} indicates an arbitrary function of $x_1 - x_2$. Substituting the expansion $\tilde{v}_{12} = \sum_{j=0}^{n}\delta_{12}^{j}\tilde{v}_{12}^{(j)}$ into Eq. (3.13) and using Eqs. (3.14) one obtains

$$\tilde{v}_{12}^{(n)} = 0; \quad \tilde{v}_{12}^{(j-1)} = (i/2)\Phi_{12}\tilde{v}_{12}^{(j)}, \quad 1 < j < n-1,$$
$$\delta_{12}q_{1_t} = (i/2)\delta_{12}\Phi_{12}\,\tilde{v}_{12}^{(0)}. \tag{3.15}$$

The iteration (3.15) implies that $\tilde{v}_{12}^{(0)} = (i/2)^{n-1} \times \Phi_{12}^{n-1}\tilde{v}_{12}^{(n-1)}$; to determine $\tilde{v}_{12}^{(n-1)}$ we notice that $\tilde{v}_{12}^{(n)} = v_{12}^{(+)n} - v_{12}^{(-)n} = 0$ implies $v_{12}^{(n)} = v_{12}^{(\pm)n} = c_n = \text{const}$, and then

$$\tilde{v}_{12}^{(n-1)} = (i/2)\,[q_{12}^{+}(v_{12}^{(+)n} - v_{12}^{(-)n})$$
$$+ q_{12}(v_{12}^{(+)n} + v_{12}^{(-)n})]c_n = ic_n q_{12}^{-} \cdot 1.$$

Equation (3.2) is then obtained defining $\beta_n \doteq i(i/2)^n c_n$.

B. Properties of the extended Hilbert transform

In this subsection we list several interesting and useful properties of the extended Hilbert transform.

Proposition 3.2: The extended Hilbert transform H_{12} enjoys the following properties.

$$(1) \quad [H_{12},h_{12}] = 0, \tag{3.16a}$$

$$(2) \quad H_{12}a(x_j) = H_j a(x_j), \quad j = 1,2, \tag{3.16b}$$

$$H_j f(x_i,x_j) \doteq \pi^{-1}\int_{\mathbf{R}} dy(y - x_j)^{-1}f(x_i,y), \quad i \neq j. \tag{3.16c}$$

$$(3) \quad \int_{\mathbf{R}} dx_2\,\delta_{12}H_{12}f_{12} = H_1 f_{11}, \tag{3.16d}$$

$$(4) \quad \partial_{x_j}H_{12}f_{12} = H_{12}\,\partial_{x_j}f_{12}, \quad j = 1,2, \tag{3.16e}$$

$$(5) \quad H_{12}^2 = -1. \tag{3.16f}$$

Moreover,

$$(6) \quad H_{12}f_{12}^{\pm}h_{12} = (H_{12}f_{12})^{\pm}h_{12}, \tag{3.17a}$$

$$(7) \quad H_{12}(g_{12}H_{12}f_{12} + (H_{12}g_{12})^{-}f_{12})$$
$$= -g_{12}f_{12} + (H_{12}g_{12})^{-}H_{12}f_{12}, \tag{3.17b}$$

$$(8) \quad H_{12}^{*} = -H_{12}. \tag{3.17c}$$

Here H_{12} induces the following analytic properties:

(9) If

$$f_{12}^{(\pm)} \doteq \pm\tfrac{1}{2}(1 \mp iH_{12})f_{12}$$
$$= (2\pi i)^{-1}\int_{\mathbf{R}} dy(y - (x_1 + x_2 \pm i0))^{-1}$$
$$\times F(y,x_1 - x_2), \tag{3.18}$$

then

(i) $f_{12}^{(+)}$ and $f_{12}^{(-)}$ are holomorphic for $\mathrm{Im}(x_1 + x_2) > 0$ and $\mathrm{Im}(x_1 + x_2) < 0$, respectively.

$$\text{(ii)} \quad f_{12}^{(+)} + f_{12}^{(-)} = -iH_{12}(f_{12}^{(+)} - f_{12}^{(-)}). \tag{3.19}$$

Proof: Equations (3.16f) and (3.17b) are interesting generalizations of well-known identities $H^2 = -1$, $H(gHf + fHg) = -gf + (Hg)(Hf)$, and can be proven using Fourier space. Equations (3.16a)–(3.16e), (3.17a), (3.17c), and (3.18), (3.19) are direct consequences of the definition of H_{12} (see Appendix B for details).

C. Algebraic properties of the BO class

In this section we show that the main algebraic properties of the BO class can be entirely described using the theory developed in Ref. 4; we refer to that paper for details and proofs.

1. Representation of the class

It was shown in Sec. III A that the BO class admits the following representation:

$$q_{1_t} = \beta_n \int_{\mathbb{R}} dx_2\, \delta_{12} \Phi_{12}^n \widehat{K}_{12}^0 \cdot 1 \doteq \beta_n \int_{\mathbb{R}} dx_2\, \delta_{12} K_{12}^{(n)} \doteq K_{11}^{(n)}, \tag{3.20}$$

where $\widehat{K}_{12}^0 = q_{12}^-$ and Φ_{12} is defined in (3.3a).

The recursion operator Φ_{12} and the "starting" operator $\widehat{K}_{12}^0$ enjoy simple commutator relations with $h_{12} = h(x_1 - x_2)$,

$$[\Phi_{12}, h_{12}] = 2i \frac{\partial h_{12}}{\partial_{x_1}}, \quad [\widehat{K}_{12}^0, h_{12}] = 0, \tag{3.21}$$

which imply that $\delta_{12} K_{12}^{(n)}$ can be written in the following *alternative form*:

$$\delta_{12} K_{12}^{(n)} = \sum_{l=0}^{n} (-2i)^l \binom{n}{l} \Phi_{12}^{n-l} \widehat{K}_{12}^0 \frac{\partial^l \delta(x_1 - x_2)}{\partial_{x_1}}. \tag{3.22}$$

2. The d derivative

As in 2 + 1 dimensions, the derivation of the extended algebraic structures of the BO class is based on integral representations of operators depending on q, ∂_x, and H. This mapping between operators and their corresponding kernels induces a mapping between derivatives and leads to the introduction of a new directional derivative, the so-called d derivative.[4] Here we briefly remark that the basic operators $q_{12}^{\pm}$ appearing in the BO formalism are the *same* as for the KP case, replacing x_j by y_j and i by the parameter α [see Eqs. (1.13b) and (1.33)]. Then their d derivative is simply given by

$$q_{12}^{\pm}[g_{12}] f_{12} \doteq g_{12}^{\pm} f_{12}, \tag{3.23}$$

$$g_{12}^{\pm} f_{12} \doteq \int_{\mathbb{R}} dx_3 (g_{13} f_{32} \pm f_{13}\, g_{32}). \tag{3.24}$$

Since Φ_{12} and $\widehat{K}_{12}^0$ are expressed in terms of $q_{12}^{\pm}$, their d derivatives are well defined,

$$\Phi_{12_d}[g_{12}] = g_{12}^+ - i g_{12}^- H_{12}, \quad \widehat{K}_{12_d}^0[g_{12}] = g_{12}^-. \tag{3.25}$$

As for the (2 + 1)-dimensional case, the connection between the d derivative and the usual Fréchet derivative is given by the following projective formula:

$$K_{12_d}[\delta_{12} g_{12}] = K_{12_d}[g] \doteq K_{12_{q_1}}[g_{11}] + K_{12_{q_2}}[g_{22}], \tag{3.26}$$

where $K_{12_{q_i}}$ denotes the Fréchet derivative of K_{12} with respect to q_i, i.e.,

$$K_{12_{q_i}}[g_{ii}] \doteq \partial_\epsilon K_{12}(q_i + \epsilon g_{ii}, q_j)|_{\epsilon=0}, \quad i,j = 1,2, \quad i \neq j. \tag{3.27}$$

3. The starting symmetry $\widehat{K}_{12}^0 \cdot h_{12}$, its Lie algebra, and its characterization through the recursion operator

The starting symmetry $K_{12}^{(0)} = q_1 - q_2$ of the BO class is written as $q_{12}^- \cdot 1$. As in 2 + 1 dimensions a crucial aspect of this theory is that the operator $\widehat{K}_{12}^0 = q_{12}^-$, acting on suitable functions $h_{12} = h(x_1 - x_2)$, solutions of the RH problem $h_{12}^{(+)} - h_{12}^{(-)} = 0$ [(+) and (−) here indicate analyticity in the upper and lower $x_1 + x_2$ half-planes, then $h_{12} = h_{12}^{(+)} = h_{12}^{(-)}$], form a Lie algebra, given by

$$[q_{12} h_{12}, q_{12} \tilde{h}_{12}]_d = -q_{12}^- [h_{12}, \tilde{h}_{12}]_l, \tag{3.28}$$

where the Lie brackets $[\ ,\ ]_d$, $[\ ,\ ]_l$ are defined by

$$[f_{12}, g_{12}]_d \doteq f_{12_d}[g_{12}] - g_{12_d}[f_{12}], \tag{3.29a}$$

$$[h_{12}, \tilde{h}_{12}]_l \doteq \int_{\mathbb{R}} dx_3 (h_{13} \tilde{h}_{32} - \tilde{h}_{13} h_{32}). \tag{3.29b}$$

As in 2 + 1 dimensions, the starting symmetry $\widehat{K}_{12}^0 \cdot h_{12}$ can be characterized through the recursion operator Φ_{12} via the equations

$$\Phi_{12}(h_{12}^{(+)} - h_{12}^{(-)}) = q_{12}^+ (h_{12}^{(+)} - h_{12}^{(-)}) + q_{12}^- (h_{12}^{(+)} + h_{12}^{(-)}) = 2\widehat{K}_{12}^0 h_{12}, \tag{3.30a}$$

$$h_{12}^{(+)} = h_{12}^{(-)} = h_{12}, \tag{3.30b}$$

obtained using Eqs. (3.3a) and (3.19).

4. Symmetries, strong and hereditary symmetries

The recursion operator Φ_{12} and the starting operator $\widehat{K}_{12}^0 = q_{12}^-$ are the ingredients of the evolution equations

$$q_{1_t} = \int_{\mathbb{R}} \delta_{12} K_{12}^{(n)}. \tag{3.31}$$

They enjoy the following properties.

Proposition 3.3: (i) The recursion operator Φ_{12} is hereditary, namely,

$$\Phi_{12_d}[\Phi_{12} f_{12}] g_{12} - \Phi_{12} \Phi_{12_d}[f_{12}] g_{12}$$

is symmetric w.r.t. f_{12} and g_{12}; (3.32)

(ii) Φ_{12} is a strong symmetry for $K_{12}^0 h_{12}$, namely,

$$\mathcal{L}(\Phi_{12}, \widehat{K}_{12}^0 h_{12}) \doteq \Phi_{12_d}[\widehat{K}_{12}^0 h_{12}] + [\Phi_{12}, (\widehat{K}_{12}^0 h_{12})_d] = 0. \tag{3.33}$$

Proof: Equations (3.32) and (3.33) are verified in Appendix A, although this check is not strictly necessary, for two reasons.

(1) Φ_{12} comes from the isospectral problem (3.1), and an extension of the theorem presented in Ref. 18 should guarantee its hereditariness (see also Ref. 4, §4.E). It is also interesting to remark that a direct proof of the hereditariness of Φ_{12} makes use of Eq. (3.17b).

(2) The hereditariness of Φ_{12} and the characterization (3.30) implies that Proposition 3.3 (ii) holds (see Lemma 4.2 of Ref. 4 and Appendix A for a direct check).

The operator Φ_{12} generates infinitely many commuting symmetries of the BO class; precisely, since Φ_{12} is a hereditary operator and strong symmetry for the starting symmetry $\widehat{K}^0_{12}h_{12}$ that satisfies Eq. (3.28), then Theorem 4.3 of Ref. 4 implies that $\sigma^{(m)}_{12} \doteq \Phi^m_{12} q^-_{12} \cdot 1$ are extended symmetries of every evolution equation of the BO class, namely,

$$\sigma^{(m)}_{12_t}[K^{(n)}] = (\delta_{12}K^{(n)}_{12})_d[\sigma^{(m)}_{12}] \tag{3.34}$$

for every non-negative integer n and m, where, using (3.22),

$$(\delta_{12}K^{(n)}_{12})_d \doteq \sum_{l=0}^{n} (-2i)^l \binom{n}{l} (\Phi^{n-l}_{12}\widehat{K}^0_{12}\delta^l_{12})_d . \tag{3.35}$$

The first three operators $(\delta_{12}K^{(n)}_{12})_d$ of the BO class are explicitly reported below:

$$(\delta_{12}K^{(0)}_{12})_d = 0, \tag{3.36a}$$

$$(\sigma_{12}K^{(1)}_{12})_d = 2i(\partial_{x_1} + \partial_{x_2}), \tag{3.36b}$$

$$\begin{aligned}(\delta_{12}K^{(2)}_{12})_d &= 4i(H_{12}(\partial_{x_1} + \partial_{x_2})^2 + (\partial_{x_1} + \partial_{x_2})(q_1 + q_2) \\ &\quad + i((H_1q_1)x_1 - (H_2q_2)x_2) \\ &\quad - i(q_1 - q_2)H_{12}(\partial_{x_1} + \partial_{x_2}))\end{aligned} \tag{3.36c}$$

(see Appendix A).

The usefulness of the extended symmetries $\sigma^{(m)}_{12}$ follows from the fact that they give rise to symmetries and Bäcklund transformations; precisely according to Theorem 4.2 of Ref. 4:

If $\sigma^{(m)}_{12}$ is an extended symmetry of Eq. (3.31), then (i) $\sigma^{(m)}_{11} = \sigma^{(m)}_{12}|_{x_2 = x_1}$ is a symmetry of Eq. (3.31), namely,

$$\sigma^{(m)}_{11_t}[K^{(n)}_{11}] = K^{(n)}_{11_t}[\sigma^{(m)}_{11}] ; \tag{3.37}$$

and (ii) the equation

$$\sigma^{(m)}_{12} = \sigma^{(m)}(q_1,q_2) = 0 \tag{3.38}$$

is a Bäcklund transformation for (3.31) where, of course, q_1 and q_2 are now viewed as two different solutions of (3.31).

5. (Bi-) Hamiltonian formalism and constants of motion in involution

Proposition 3.4: (i) If we define

$$\Theta^{(1)}_{12} \doteq q^-_{12}, \quad \Theta^{(2)}_{12} \doteq \Phi_{12}\Theta^{(1)}_{12}, \tag{3.39}$$

then $\Theta_{12} \doteq \Theta^{(1)}_{12} + \kappa\Theta^{(2)}_{12}$ is a Hamiltonian operator for all constants κ, namely,

(a) $\Theta^*_{12} = -\Theta_{12}$, (3.40a)

(b) Θ_{12} satisfy the Jacobi identity w.r.t. the bracket

$$\{a_{12},b_{12},c_{12}\} \doteq \langle a_{12},\Theta_{12_d}[\Theta_{12}b_{12}]c_{12}\rangle . \tag{3.40b}$$

(ii) The adjoint Φ^*_{12} of the recursion operator, given by

$$\Phi^*_{12} = q^+_{12} - iH_{12}q^-_{12} , \tag{3.41}$$

satisfies the following "well-coupling" condition:

$$\Phi_{12}\Theta^{(1)}_{12} = \Theta^{(1)}_{12}\Phi^*_{12} . \tag{3.42}$$

(iii) $\hat{\gamma}^{(1)}_{12} \cdot h_{12} = \Phi^*_{12} \cdot h_{12}$ is an extended gradient, namely,

$$(\hat{\gamma}^{(1)}_{12}h_{12})_d = (\hat{\gamma}^{(1)}_{12}h_{12})^*_d . \tag{3.43}$$

Proof: Equations (3.40)–(3.42) are a direct consequence of the definitions (3.39), of Eqs. (3.17b) and (3.17c), and of the property $q^{\pm\,*}_{12} = \pm q^{\pm}_{12}$.

Remark 3.3: Using Eq. (3.42) the BO class can be written in the following form:

$$\begin{aligned} q_{1_t} &= \beta_n \int dx_2\, \delta_{12}q^-_{12}(\Phi^*_{12})^n \cdot 1 \\ &= \beta_n \int dx_2\, q^-_{12}\delta_{12}(\Phi^*_{12})^n \cdot 1 \\ &= i\,\partial_{x_1} \int dx_2\, \delta_{12}(\Phi^*_{12})^n \cdot 1 = i\beta_n\, \partial_{x_1}\gamma^{(n)}_{11} . \end{aligned} \tag{3.44}$$

The first Hamiltonian operator $\Theta^{(1)}_{12} = q^-_{12}$ commutes with δ_{12} and reduces to $i\,\partial_{x_1}$. Then ∂_x is the (projected version of the) first Hamiltonian operator of the BO class; this result was already known.[15]

The existence of a compatible pair of Hamiltonian operators is connected to the existence of infinitely many constants of motion in involution. Theorems 4.1–4.5 of Ref. 4 can finally be summarized in the following proposition.

Proposition 3.5: Consider the compatible pair of Hamiltonian operators $\Theta^{(1)}_{12} \doteq q^-_{12}$, $\Theta^{(2)}_{12} \doteq (q^+_{12} - iq^-_{12}H_{12})q^-_{12}$ and define $\Phi_{12} \doteq \Theta^{(2)}_{12}(\Theta^{(1)}_{12})^{-1}$; then the following is true.

(i) Φ_{12} is a hereditary operator.

(ii) $\sigma^{(m)}_{12} \doteq \Phi^{(m)}_{12}q^-_{12} \cdot 1$ and $\gamma^{(m)}_{12} \doteq (\Phi^*_{12})^m \cdot 1$ are extended symmetries and extended gradients of conserved quantities, respectively, for Eqs. (3.2), namely,

$$\sigma^{(m)}_{12_t}[K^{(n)}] = (\delta_{12}K^{(n)}_{12})_d[\sigma^{(m)}_{12}] , \tag{3.45a}$$

$$\gamma^{(m)}_{12_t}[K^{(n)}] + (\delta_{12}K^{(n)}_{12})^*_d[\gamma^{(m)}_{12}] = 0 , \tag{3.45b}$$

$$((\Phi^*_{12})^m h_{12})_d = ((\Phi^*_{12})^m h_{12})^*_d, \quad h_{12} = h(x_1 - x_2) . \tag{3.45c}$$

(iii) Equations (3.2) are bi-Hamiltonian systems, since they can be written in the following two "extended" Hamiltonian forms

$$q_{1_t} = \beta_n \int_{\mathbb{R}} dx_2\, \delta_{12}\Theta^{(1)}_{12}\gamma^{(n)}_{12} = \beta_n \int_{\mathbb{R}} dx_2\, \delta_{12}\Theta^{(2)}_{12}\gamma^{(n-1)}_{12} . \tag{3.46}$$

(iv) $\sigma^{(m)}_{11}$ and $\gamma^{(m)}_{11}$ are symmetries and gradients of conserved quantities for Eq. (3.2), namely,

$$\sigma^{(m)}_{11_t}[K^{(n)}_{11}] = K^{(n)}_{11_t}[\sigma^{(m)}_{11}] , \tag{3.47a}$$

$$\gamma^{(m)}_{11_t}[K^{(n)}_{11}] + K^{(n)}_{11_t}[\sigma^{(m)}_{11}] = 0 , \tag{3.47b}$$

$$\gamma^{(m)}_{11_t} = \gamma^{(m)\,+}_{11_t} , \tag{3.47c}$$

where $^+$ denotes the operation of adjoint w.r.t. the bilinear form $(f,g) \doteq \int_{\mathbb{R}} dx\, fg$.

(v) The corresponding conserved quantities I_m, related to $\gamma^{(m)}_{12}$ and $\gamma^{(m)}_{11}$ via equations

$$\gamma_{12}^{(m)} = \mathrm{grad}_{12}\, I_m, \quad I_{m_d}[f_{12}] \doteq \langle \mathrm{grad}_{12}\, I_m, f_{12}\rangle, \tag{3.48a}$$

$$\gamma_{11}^{(m)} = \mathrm{grad}\, I_m, \quad I_m[f] \doteq (\mathrm{grad}\, I_m, f), \tag{3.48b}$$

are constants of motion of Eqs. (3.2).

(vi) These constants of motion are in involution with respect to the Poisson brackets

$$\{I_n, I_m\} \doteq \langle \delta_{12}\gamma_{12}^{(n)}, \Theta_{12}\gamma_{12}^{(m)}\rangle, \quad \Theta_{12} = \Theta_{12}^{(1)} \text{ and/or } \Theta_{12}^{(2)}, \tag{3.49}$$

namely,

$$\{I_n, I_m\} = 0. \tag{3.50}$$

(vii) The equations $K_{12}^{(m)} = K^{(m)}(q_1, q_2) = 0$ are Bäcklund transformations (BT) for the BO class (3.2), interpreting q_1 and q_2 as to different solutions of (3.2).

Remark 3.4: (i) The first extended symmetries of the BO class are given by

$$\sigma_{12}^{(0)} = q_{12}^- \cdot 1 = q_1 - q_2, \tag{3.51a}$$

$$\begin{aligned}\sigma_{12}^{(1)} &= \Phi_{12}^{(1)} q_{12}^- \cdot 1 \\ &= i(q_{1_{x_1}} + q_{2_{x_2}}) + H_1 q_{1_{x_1}} - H_2 q_{2_{x_2}} \\ &\quad + (q_1 + q_2)(q_1 - q_2) \\ &\quad - i(q_1 - q_2)(H_1 q_1 - H_2 q_2),\end{aligned} \tag{3.51b}$$

then their projections are the first symmetries of the BO class

$$\sigma_{11}^{(0)} = 0, \quad \sigma_{11}^{(1)} = 2iq_{1_{x_1}}, \tag{3.52}$$

and equations

$$\sigma_{12}^{(0)} = 0, \quad \sigma_{12}^{(1)} = 0, \tag{3.53}$$

are the first two BT's of the class. We remark that the BT's generated by Φ_{12} are polynomial in q_1, q_2, unlike the previously known examples.[17]

D. Connection with the mastersymmetries theory

The mastersymmetry approach was introduced by Fuchssteiner and one of the authors (A.S.F.)[15] as an alternative way of generating symmetries of the BO equation. This approach was subsequently applied to (2 + 1)-dimensional systems like KP,[11] 1 + 1 systems like KdV,[12,18] and finite-dimensional systems like the Calogero–Moser problem.[19]

In this section we briefly show that the existence of a hereditary operator Φ_{12} allows a simple and elegant characterization of the BO mastersymmetries (analogous and more detailed results for KP were reported in Ref. 5).

Proposition 3.6: (i) If

$$K_{12}^{(n)} \doteq \Phi_{12}^n q_{12}^- \cdot 1, \tag{3.54a}$$

$$\tau_{12}^{(m,r)} \doteq \Phi_{12}^m q_{12}^- \cdot (x_1 + x_2)^r, \tag{3.54b}$$

then

$$[\delta_{12} K_{12}^{(n)}, \tau_{12}^{(m,1)}]_d = 4inK_{12}^{(n+m-1)}. \tag{3.55}$$

(ii) $\tau_{11}^{(m,1)} \doteq \tau_{12}^{(m,1)}|_{x_2=x_1}$ are mastersymmetries of degree 1 of the BO class, since

$$[K_{11}^{(n)}, \tau_{11}^{(m,1)}]_f = 4inK_{11}^{(n+m-1)}. \tag{3.56}$$

Proof: The derivation of Eq. (3.55), presented in Appendix C, is based on the following important properties:

$$\begin{aligned}(1)\ \ &\Phi_{12_d}[q_{12}(x_1 + x_2)] + [\Phi_{12}, (q_{12}(x_1 + x_2))_d] \\ &= iq_{12}[H_{12}, (x_1 + x_2)\cdot\ ],\end{aligned} \tag{3.57a}$$

$$(2)\ \ [H_{12}, (x_1 + x_2)]f_{12} = \frac{1}{\pi}\int_{\mathbf{R}} dy\, F(y, x_1 - x_2), \tag{3.57b}$$

$$f_{12} = f(x_1, x_2) = F(x_1 + x_2, x_1 - x_2), \tag{3.57c}$$

$$(3)\ \ iq_{12}[H_{12}, (x_1 - x_2)\cdot\ ](\delta_{12}^s K_{12}^{(l)}) = 0, \quad \forall s, l \geq 0. \tag{3.57d}$$

These follow from the definitions (3.3) and from equation

$$\lim_{|x_i| \to \infty} \left(\sum_{l=0}^{s-1} (-1)^{s-1} \partial_{x_1}^{s-l-1} \partial_{x_2}^{l} K_{12}^{(n)}\right)_{x_1 = x_2} = 0 \tag{3.57e}$$

(see Appendix C). Equation (3.56) follows from (3.55) using Theorem 4.1 of Ref. 4.

Remark 3.4: As for the KP case,[5] time-dependent symmetries of the BO hierarchy should be generated via mastersymmetries $\tau_{12}^{(m,r)}$ of degree $r > 1$. In this case, an equation analogous to (3.55) should follow from a suitable generalization of Eq. (3.57a) obtained replacing $(x_1 + x_2)$ by $(x_1 + x_2)^r$, $r > 1$.

E. Connection with the complex Burgers hierarchy

It is well known that if $q(x,t)$ is analytic in the upper x plane, then the BO equation (1.30) reduces to the (complex) Burgers equation

$$q_t = 2qq_x + iq_{xx}, \tag{3.58}$$

since

$$Hf^{(\pm)} = \pm if^{(\pm)}, \tag{3.59}$$

where $f^{(+)}(x)$ and $f^{(-)}(x)$ are holomorphic in the upper and lower half x plane, respectively. The same result obviously holds for the whole hierarchy.

Proposition 3.7: If $q(x,t)$ is holomorphic in the upper x plane, then the BO hierarchy (3.2) reduces to the following complex Burgers hierarchy (investigated in Ref. 20):

$$q_t = b_n(i\,\partial_x + \partial_x q\,\partial_x^{-1})^{n-1} q_x, \quad n \geq 1, \tag{3.60a}$$

$$b_n \doteq 2^n i\beta_n, \quad \partial_x^{-1} \doteq \int_{-\infty}^{x} dx. \tag{3.60b}$$

Proof: The proof is straightforward and relies on the fact that each gradient $\gamma_{12}^{(n)}$ is a holomorphic function in the upper x_1 and x_2 planes; hence Eq. (3.59) implies that

$$\begin{aligned}\Phi_{12}^* \gamma_{12}^{(n)} &= (q_{12}^+ - iH_{12} q_{12}^-)\gamma_{12}^{(n)} \\ &= (q_{12}^+ + q_{12}^-)\gamma_{12}^{(n)} = 2(q_1 + i\,\partial_{x_1})\gamma_{12}^{(n)}.\end{aligned}$$

Then

$$\begin{aligned}q_{1_t} &= \beta_n \int_{\mathbf{R}} dx_2\, \delta_{12} q_{12}^- (\Phi_{12}^*)^n \cdot 1 \\ &= 2i\beta_n\, \partial_{x_1} \int_{\mathbf{R}} dx_2\, \delta_{12} (\Phi_{12}^*)^n \cdot 1 \\ &= 2^{n+1}\beta_n\, \partial_{x_1}(q_1 + i\,\partial_{x_1})^n \cdot 1\end{aligned}$$

$$= b_n \partial_{x_1} (q_1 + i \partial_{x_1})^{n-1} q_1$$
$$= b_n \partial_{x_1} (q_1 + i \partial_{x_1})^{n-1} \partial_{x_1}^{-1} q_{1_{x_1}}$$
$$= b_n (i \partial_{x_1} + \partial_{x_1} q_1 \partial_{x_1}^{-1})^{n-1} q_{1_{x_1}} .$$

ACKNOWLEDGMENTS

It is a pleasure to acknowledge useful discussions with M. J. Ablowitz, O. Ragnisco, and M. Bruschi. One of the authors (P.M.S.) wishes to thank the friendly hospitality of the Department of Mathematics and Computer Science of Clarkson University.

This work was partially supported by the Office of Naval Research under Grant No. N00014-76-C-0867 and the National Science Foundation under Grant No. DMS-8501325.

APPENDIX A

In this appendix we use the notion of directional derivative and extended bilinear form introduced in (1.24) and (1.22), (1.32), respectively, to prove some of the results presented in this paper. In order to give a self-contained presentation, we first present some results contained in Appendix C of Ref. 4.

The directional derivative of the basic operators $q_{12}^{\pm}$ (1.13b), (1.23), (1.33), is

$$q_{12_d}^{\pm}[f_{12}]g_{12} = f_{12}^{\pm} g_{12} , \tag{A1}$$

where the integral operators $f_{12}^{\pm}$, defined by

$$f_{12}^{\pm} g_{12} \doteq \int_{\mathbf{R}} dx_3 (f_{13} g_{32} \pm g_{13} f_{32}) , \tag{A2}$$

enjoy the following algebraic properties:

$$a_{12}^{\pm} b_{12} = \pm b_{12}^{\pm} a_{12} , \tag{A3a}$$

$$(a_{12}^{\pm} b_{12}^{\pm} - b_{12}^{\pm} a_{12}^{\pm}) c_{12} = (a_{12} b_{12})^{-} c_{12} = - c_{12}^{-} a_{12} b_{12} , \tag{A3b}$$

$$(a_{12}^{\pm} b_{12} \mp b_{12}^{\pm} a_{12}) c_{12} = (a_{12}^{\pm} b_{12})^{-} c_{12} + \pm c_{12}^{\pm} a_{12}^{\pm} b_{12} , \tag{A3c}$$

$$a_{12}^{\pm *} = \pm a_{12}^{\pm} . \tag{A3d}$$

Moreover the integral representation

$$q_{12}^{\pm} f_{12} = \int_{\mathbf{R}} dx_3 (q_{13} f_{32} \pm f_{13} q_{32}) \tag{A4}$$

implies that $q_{12}^{\pm}$ satisfy Eqs. (A3) as well. Equations (A3) are conveniently used to prove the following properties of the recursion and Hamiltonian operators of the KP and BO equations.

For the KP class, the following is true.

(1) $\Phi_{12} \doteq (\alpha \partial_y + q_{12}^{-})(D_1 + D_2)^{-1}$ is a strong symmetry of $\hat{K}_{12}^0 H_{12} = (\alpha \partial_y + q_{12}^{-}) H_{12}$. Indeed

$$\Phi_{12_d}[\sigma_{12}] = \sigma_{12}^{-} (D_1 + D_2)^{-1} ,$$
$$(\hat{K}_{12}^0 H_{12})_d [\sigma_{12}] = \sigma_{12}^{-} H_{12} ,$$

and

$$\mathscr{L}'(\Phi_{12}, \hat{K}_{12}^0 H_{12}) f_{12}$$
$$= (\hat{K}_{12}^0 H_{12})^{-} (D_1 + D_2)^{-1} f_{12}$$
$$- (\Phi_{12} f_{12})^{-} H_{12} + \Phi_{12} f_{12}^{-} H_{12}$$
$$= ((\alpha \partial_y + q_{12}^{-}) H_{12})^{-} g_{12}$$
$$- ((\alpha \partial_y + q_{12}^{-}) g_{12})^{-} H_{12} + (\alpha \partial_y + q_{12}^{-}) g_{12}^{-} H_{12},$$

having introduced $g_{12} \doteq (D_1 + D_2)^{-1} f_{12}$ and used $H_{12}(D_1 + D_2)^{-1} = (D_1 + D_2)^{-1} H_{12}$. Using (A3a) we obtain $g_{12}^{-} q_{12} H_{12} - H_{12}^{-} q_{12} g_{12} - q_{12} g_{12}^{-} H_{12}$, which is zero, for (A3b)$^{-}$.

(2) Φ_{12} is a hereditary operator. Indeed

$$\Phi_{12_d}[\Phi_{12} f_{12}] g_{12} - \Phi_{12} \Phi_{12_d}[f_{12}] g_{12} - (\text{sym. w.r.t. } f_{12} \leftrightarrow g_{12})$$
$$= ((\alpha \partial_y + q_{12}^{-})(D_1 + D_2)^{-1} f_{12})^{-} (D_1 + D_2)^{-1} g_{12} - (\alpha \partial_y + q_{12}^{-})(D_1 + D_2)^{-1} f_{12}^{-} (D_1 + D_2)^{-1} g_{12} - (\text{sym.} \cdots)$$
$$= ((D_1 + D_2)^{-1} g_{12})^{-} (\alpha \partial_y + q_{12}^{-})(D_1 + D_2)^{-1} f_{12} - ((D_1 + D_2)^{-1} f_{12})^{-} (\alpha \partial_y + q_{12}^{-})(D_1 + D_2)^{-1} g_{12}$$
$$- (\alpha \partial_y + q_{12}^{-})(D_1 + D_2)^{-1} (f_{12}^{-} (D_1 + D_1)^{-1} g_{12} - g_{12}^{-} (D_1 + D_2)^{-1} f_{12}) = 0 ,$$

using integration by parts,

$$(D_1 + D_2)^{-1} f_{12}^{-} (D_1 + D_2)^{-1} g_{12}^{-}$$
$$= ((D_1 + D_2)^{-1} f_{12})^{-} (D_1 + D_2)^{-1} g_{12} - (D_1 + D_2)^{-1}$$
$$\times (((D_1 + D_2)^{-1} f_{12})^{-} g_{12} - g_{12}^{-} (D_1 + D_2)^{-1} f_{12})$$

and Eq. (A3b).

(3) $[\hat{K}_{12}^0 H_{12}^{(1)}, \hat{K}_{12}^0 H_{12}^{(2)}]_d$

$$= (\hat{K}_{12}^0 H_{12}^{(2)})^{-} H_{12}^{(1)} - (\hat{K}_{12}^0 H_{12}^{(1)})^{-} H_{12}^{(2)}$$
$$= - H_{12}^{(1)-} (\alpha \partial_y + q_{12}^{-})$$
$$+ H_{12}^{(2)-} (\alpha \partial_y + q_{12}^{-}) H_{12}^{(1)}$$
$$= - \hat{K}_{12}^0 H_{12}^{(1)-} H_{12}^{(2)} ,$$

for (A3a)$^{-}$ and (A3b)$^{-}$.

For the BO class the following is true.

(4) Φ_{12} is a strong symmetry of $q_{12}^{-} h_{12}$, $h_{12} = h(x_1 - x_2)$. Indeed, using (3.43a), we have

$$\mathscr{L}'(\Phi_{12}, q_{12} h_{12}) = (q_{12} h_{12})^{+} f_{12} - i(q_{12}^{-} h_{12})^{-} H_{12} f_{12}$$
$$+ (q_{12}^{+} - i q_{12}^{-} H_{12}) f_{12}^{-} h_{12}$$
$$- (q_{12}^{+} f_{12} - i q_{12}^{-} H_{12} f_{12})^{-} h_{12} .$$

Using Eqs. (A3a) and property (3.17a) [see Appendix B (5)] we obtain

$$(f_{12}^{+} q_{12}^{-} h_{12} + q_{12}^{+} f_{12} h_{12} + h_{12}^{-} q_{12}^{+} f_{12})$$
$$+ i((H_{12} f_{12})^{-} q_{12}^{-} h_{12} - q_{12}^{-} (H_{12} f_{12})^{-} h_{12}$$
$$- h_{12}^{-} q_{12} H_{12} f_{12}) ,$$

and the two expressions in parentheses are zero using (A3c) and (A3b)$^-$, respectively.

(5) Φ_{12} is a hereditary operator. Using (3.24a) we have that

$$\Phi_{12_d}[\Phi_{12}f_{12}]g_{12}-\Phi_{12}\Phi_{12_d}[f_{12}]g_{12}$$
$$-(\text{sym. w.r.t. } f_{12}\leftrightarrow g_{12})$$
$$=(q_{12}^+f_{12}-iq_{12}^-H_{12}f_{12})g_{12}^+$$
$$-i(q_{12}^+f_{12}-iq_{12}^-H_{12}f_{12})^-H_{12}\,g_{12}$$
$$-q_{12}^+(f_{12}^+g_{12}-if_{12}^-H_{12}\,g_{12})$$
$$+iq_{12}^-H_{12}(f_{12}^+g_{12}-if_{12}^-H_{12}\,g_{12})$$
$$-(\text{sym. w.r.t. } f_{12}\leftrightarrow g_{12}).$$

Using (A3c) and (A3b) we obtain

$$q_{12}^-(H_{12}(g_{12}^-H_{12}f_{12}+(H_{12}\,g_{12})^-f_{12})$$
$$+g_{12}^-f_{12}-(H_{12}\,g_{12})^-H_{12}\,f_{12}),$$

which is zero for Eq. (3.17b).

(6) $q_{12}^{\pm *}=\pm q_{12}^{\pm}$.

These are direct consequences of the definitions (1.32) and (1.33). Their immediate implications are Eqs. (3.17c), (3.40a), and (3.41).

(7) $\Theta_{12}^{(1)}=q_{12}^-$ and $\Theta_{12}^{(2)}$ are Hamiltonian operators. They are skew symmetric, since

$$\Theta_{12}^{(2)*}=((q_{12}^+-iq_{12}^-H_{12})q_{12}^-)^*=q_{12}^{-*}(q_{12}^{+*}-iH_{12}^*q_{12}^{-*})$$
$$=-q_{12}(q_{12}^+-iH_{12}q_{12}^-)$$
$$=-\Theta_{12}^{(2)}\ (\text{being } q_{12}^+q_{12}=q_{12}q_{12}^+).$$

They satisfy the Jacobi identity (3.40b), for instance

$$\langle a_{12},\Theta_{12_d}^{(1)}[\Theta_{12}^{(1)}b_{12}]c_{12}\rangle$$
$$=\langle a_{12},q_{12_d}^-[q_{12}^-b_{12}]c_{12}\rangle+\text{cycl. perm. } s$$
$$=\langle a_{12},(q_{12}^-b_{12})^-c_{12}\rangle+\text{cycl. perm. } s.$$

Using (A3a) and (A3d) we obtain

$$\langle a_{12},-c_{12}^-q_{12}^-b_{12}+b_{12}^-q_{12}^-c_{12}-q_{12}^-b_{12}^-c_{12}\rangle,$$

which is zero for any a_{12}, b_{12}, c_{12}, for Eq. (A3b).

(8) The derivation of Eqs. (3.36) is the same as for the corresponding ones of the KP hierarchy (see Appendix C of Ref. 4) and makes extensive use of the equations

$$(\delta_{12}^n)^{\pm}f_{12}=(\partial_{x_1}^n\pm(-1)^n\partial_{x_2}^n)f_{12}, \tag{A5}$$
$$(\delta_{12}K_{12}^{(0)})_d[f_{12}]=(\delta_{12}q_{12}^-\cdot 1)_d[F_{12}]=(q_{12}^-\delta_{12})_d[f_{12}]=f_{12}^-\delta_{12}=-\delta_{12}^-f_{12}=0,$$
$$(\delta_{12}K_{12}^{(1)})_d[f_{12}]=(\Phi_{12}q_{12}^-\delta_{12})_d[f_{12}]-2i(q_{12}^-\delta_{12}^1)_d[f_{12}]$$
$$=\Phi_{12_d}[f_{12}]q_{12}^-\delta_{12}+\Phi_{12}q_{12_d}^-[f_{12}]\delta_{12}-2iq_{12_d}^-[f_{12}]\delta_{12}^1$$
$$=(f_{12}^+-if_{12}^-H_{12})q_{12}^-\delta_{12}+\Phi_{12}f_{12}^-\delta_{12}-2if_{12}^-\delta_{12}^1$$
$$=f_{12}^+q_{12}^-\delta_{12}-if_{12}^-H_{12}q_{12}^-\delta_{12}-\Phi_{12}\delta_{12}^-f_{12}+2i(\delta_{12}^1)^-f_{12}=2i(\partial_{x_1}+\partial_{x_2})f_{12},$$

since

$$f_{12}^+q_{12}^-\delta_{12}=q_{12}^-f_{12}^+\delta_{12}-\delta^+q_{12}^-f_{12}=2(q_{12}^--q_{12}^-)f_{12}=0,$$
$$f_{12}^-H_{12}q_{12}^-\delta_{12}=f_{12}^-\delta_{12}H_{12}(q_1-q_2)=f_{12}^-\delta_{12}(H_1q_1-H_2q_2)=0,$$
$$(\delta_{12}^1)^-f_{12}=(\partial_{x_1}+\partial_{x_2})f_{12},$$
$$(\delta_{12}K_{12}^{(2)})_d[f_{12}]=(\Phi_{12}^2q_{12}^-\delta_{12})_d[f_{12}]-4i(\Phi_{12}q_{12}^-\delta_{12}^1)_d[f_{12}]-4(q_{12}^-\delta_{12}^2)_d[f_{12}]$$
$$=\Phi_{12_d}[f_{12}]\Phi_{12}q_{12}^-\delta_{12}+\Phi_{12}\Phi_{12_d}[f_{12}]q_{12}^-\delta_{12}+\Phi_{12}^2q_{12_d}[f_{12}]\delta_{12}$$
$$-4i\Phi_{12_d}[f_{12}]q_{12}^-\delta_{12}^1-4i\Phi_{12}q_{12_d}^-[f_{12}]\delta_{12}^1-4q_{12_d}^-[f_{12}]\delta_{12}^2$$
$$=(f_{12}^+-if_{12}^-H_{12})\Phi_{12}q_{12}^-\delta_{12}+\Phi_{12}(f_{12}^+-if_{12}^-H_{12})q_{12}^-\delta_{12}+\Phi_{12}^2f_{12}^-\delta_{12}$$
$$-4i(f_{12}^+-if_{12}^-H_{12})q_{12}^-\delta_{12}^1-4i\Phi_{12}f_{12}^-\delta_{12}^1-4f_{12}^-\delta_{12}^2$$
$$=4i(H_{12}(\partial_{x_1}+\partial_{x_2})^2+(\partial_{x_1}+\partial_{x_2})(q_1+q_2)+i(H_1q_{1_{x_1}}-H_2q_{2_{x_2}})-i(q_1-q_2)H_{12}(\partial_{x_1}+\partial_{x_2})),$$

since, for instance,

$$f_{12}^+\Phi_{12}q_{12}^-\delta_{12}=f_{12}^+(\delta_{12}K_{12}^{(1)}+2i\delta_{12}^1K_{12}^{(0)})=f_{12}(K_{22}^{(1)}+K_{11}^{(1)})-2i[(\partial_{x_1}(K_{32}^{(0)}f_{13}))_{x_3=x_2}-(\partial_{x_3}(K_{13}^{(0)}f_{32}))_{x_3=x_1}]$$
$$=f_{12}(2i(q_{1_{x_1}}+q_{2_{x_2}})-2i(q_{1_{x_1}}+q_{2_{x_2}}))=0,$$
$$f_{12}^-H_{12}(\delta_{12}K_{12}^{(1)}+2i\delta_{12}^1K_{12}^{(0)})=f_{12}^-(\delta_{12}H_{12}K_{12}^{(1)}+2i\delta_{12}^1H_{12}K_{12}^{(0)})$$
$$=f_{12}(H_2K_{22}^{(1)}-H_1K_{11}^{(1)})-2i[(\partial_{x_3}(f_{13}H_{32}K_{32}^{(0)}))_{x_3=x_2}+(\partial_{x_3}(f_{32}H_{13}K_{13}^{(0)}))_{x_3=x_1}]$$
$$=2i((H_1q_{1_{x_1}}-H_2q_{2_{x_2}})-(H_1q_{1_{x_1}}-H_2q_{2_{x_2}}))f_{12}=0;$$
$$f_{12}^+q_{12}^-\delta_{12}^1=q_{12}^-f_{12}^+\delta_{12}^1-\delta^{1+}q_{12}^-f_{12}=(q_{12}^-(\partial_{x_1}-\partial_{x_2})-(\partial_{x_1}-\partial_{x_2})q_{12}^-)f_{12}=-(q_{1_{x_1}}+q_{2_{x_2}})f_{12};$$
$$f_{12}^-H_{12}q_{12}^-\delta_{12}^1=f_{12}^-\delta_{12}^1(H_1q_1-H_2q_2)=-(\partial_{x_3}(f_{13}(H_3q_3-H_2q_2)))_{x_3=x_2}-(\partial_{x_3}f_{32}(H_1q_1-H_3q_3))_{x_3=x_1}$$
$$=-H_2q_{2_{x_2}}+H_1q_{1_{x_1}}.$$

APPENDIX B

In this appendix we prove some of the properties of the extended Hilbert transform presented in Proposition 3.2.

$$(1)\ \int_{\mathbb{R}} dx_2\, \delta_{12} H_{12}\, g_{12} = H_1 g_{11}\,,$$

since

$$\int_{\mathbb{R}} dx_2\, \delta_{12} \pi^{-1} \int_{\mathbb{R}} dy [y - (x_1 + x_2)]^{-1} G(y, x_1 - x_2)$$

$$= \pi^{-1} \int_{\mathbb{R}} dy (y - 2x_1)^{-1} G(y,0)$$

$$= \pi^{-1} \int_{\mathbb{R}} dy (y - x_1)^{-1} G(2y,0) = H_1 g_{11}\,,$$

$$g(x_1,x_2) \doteq G(x_1 + x_2, x_1 - x_2)\,.$$

$$(2)\ H_{12} a(x_j) = H_j a(x_j),\quad j = 1,2\,,$$

since

$$H_{12} a(x_1)$$

$$= \pi^{-1} \int_{\mathbb{R}} dy [y - (x_1 + x_2)]^{-1} a\left(\frac{y}{2} + \frac{x_1 - x_2}{2}\right)$$

$$= \pi^{-1} \int_{\mathbb{R}} dy (y - x_1)^{-1} a(y) = H_1 a(x_1)\,.$$

APPENDIX C

In order to prove that Eq. (3.55) holds, we must first derive Eqs. (3.57).

(a) Derivation of Eqs. (3.57):

$$\mathscr{L}(\Phi_{12}, q_{12}^-(x_1 + x_2)) f_{12}$$

$$= (q_{12}^-(x_1 + x_2))^+ f_{12} - i(q_{12}^-(x_1 + x_2))^- H_{12} f_{12}$$

$$+ (q_{12}^+ - i q_{12}^- H_{12}) f_{12}^-(x_1 + x_2)$$

$$- (q_{12}^+ f_{12} - i q_{12}^- H_{12} f_{12})^-(x_1 + x_2)\,.$$

Then, using Eqs. (A3a), (A3c), and (A3b), we obtain

$$\mathscr{L}(\Phi_{12}, q_{12}^-(x_1 + x_2)) f_{12}$$

$$= i q_{12}^- ((H_{12} f_{12})^-(x_1 + x_2) - H_{12}\, f_{12}^-(x_1 + x_2))$$

$$= i q_{12}^- (H_{12}(x_1 + x_2)^- - (x_1 + x_2)^- H_{12})f_{12}\,,$$

which is Eq. (3.57a)

Equation (3.57b) is a straightforward generalization of equation

$$[H,x] f = \frac{1}{\pi} \int_{\mathbb{R}} dx' f(x')\,.$$

In order to prove Eq. (3.57d), we first prove that

$$(H_{12}(x_1 + x_2)^- - (x_1 + x_2)^- H_{12})(\delta_{12}^s f_{12}) = c_s\,, \quad \text{(C1a)}$$

$$c_s \doteq \frac{1}{\pi} \int_{\mathbb{R}^2} dx_1\, dx_2\, \delta_{12} (\partial_{x_1} + \partial_{x_2})$$

$$\times \sum_{l=0}^{s-1} (-1)^{s-l} \partial_{x_1}^{s-l-1} \partial_{x_2}^{l} f_{12}$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} dx_2 \partial_{x_2} \left(\sum_{l=0}^{s-1} (-1)^{s-l} \partial_{x_1}^{s-l-1} \partial_{x_2}^{l} f_{12} \right)_{x_1 = x_2}, \quad \text{(C1b)}$$

$$(H_{12}(x_1 + x_2)^- - (x_1 + x_2)\ H_{12}) \delta_{12}^s f_{12}$$

$$= H_{12} \int_{\mathbb{R}} dx_3 [(x_1 + x_3)\delta_{32}^s f_{32} - \delta_{13}^s f_{13}(x_3 + x_2)]$$

$$- \int_{\mathbb{R}} dx_3 [(x_1 + x_3) H_{32} \delta_{32}^s f_{32}$$

$$- (H_{13} \delta_{13}^s f_{13})(x_3 + x_2)]$$

$$= H_{12}((-1)^s (s(\partial_{x_1}^{s-1} f_{32})_{x_1 = x_2}$$

$$+ (x_1 + x_2)(\partial_{x}^{s} f_{32})_{x_1 = x_2})$$

$$- s(\partial_{x_1}^{s-1} f_{13})_{x_1 = x_1} - (x_1 + x_2)(\partial_{x}^{s} f_{13})_{x_1 = x_1})$$

$$- (-1)^s (s(H_{32} \partial_{x_1}^{s-1} f_{32})_{x_1 = x_2}$$

$$+ (x_1 + x_2)(H_{32} \partial_{x}^{s} f_{32})_{x_1 = x_2}) + s(H_{13} \partial_{x_1}^{s-1} f_{13})_{x_1 = x_1}$$

$$+ (x_1 + x_2)(H_{13} \partial_{x}^{s} f_{13})_{x_1 = x_1}\,,$$

where we have used Eq. (3.16e); using now (3.16d) we obtain

$$[H_{12}, (x_1 + x_2)]((-1)^s (\partial_{x}^{s} f_{32})_{x_1 = x_2} - (\partial_{x}^{s} f_{13})_{x_1 = x_1}),$$

and Eq. (3.57b) finally leads to Eq. (C1).

Equation (3.57d) directly follows from Eq. (C1) when $f_{12} = K_{12}^{(l)}$, since Eq. (3.57e) holds.

(b) Derivation of Eq. (3.55):

$$[\delta_{12} K_{12}^{(n)}, \tau_{12}^{(m,1)}]_d$$

$$= \sum_{l=0}^{n} (-2i)^l \binom{n}{l} [\Phi_{12}^{n-l} q_{12}^- \delta_{12}^l, \Phi_{12}^m q_{12}^-(x_1 + x_2)]_d$$

$$= \sum_{l=0}^{n} (-2i)^l \binom{n}{l} \Big(\Phi_{12}^{n+m-l} [q_{12}^- \delta_{12}^l, q_{12}^-(x_1 + x_2)]_d$$

$$+ i\Phi_{12}^m \sum_{r=1}^{n-l} \Phi_{12}^{n-l-r} q_{12}^- [H_{12}, (x_1 + x_2)^-]$$

$$\times \Phi_{12}^{r-1} q_{12}^- \delta_{12}^l \Big),$$

having used the fact that Φ_{12} is a strong symmetry of $q_{12}^- h_{12}$, Eq. (3.57a) and Eq. (2.8) of Ref. 5. Equation (3.28) and equation $[\delta_{12}^l, (x_1 + x_2)]_l = 2\delta_{1,l}$, $\delta_{1,l} = 1$ if $l = 1$ and 0 if $l \neq 1$, then yield

$$4inK_{12}^{(n+m-1)}$$

$$+ i \sum_{l=0}^{n} \sum_{r=1}^{n-l} \sum_{j=0}^{r-1} (-2i)^l \binom{n}{l} (2i)^j \binom{r-1}{j} \Phi_{12}^{n-l+m-r} q_{12}^-$$

$$\times [H_{12}, (x_1 + x_2)^-](\delta_{12}^{l+j} K_{12}^{(r-1-j)}) = 4inK_{12}^{(n+m-1)},$$

for Eq. (3.57d).

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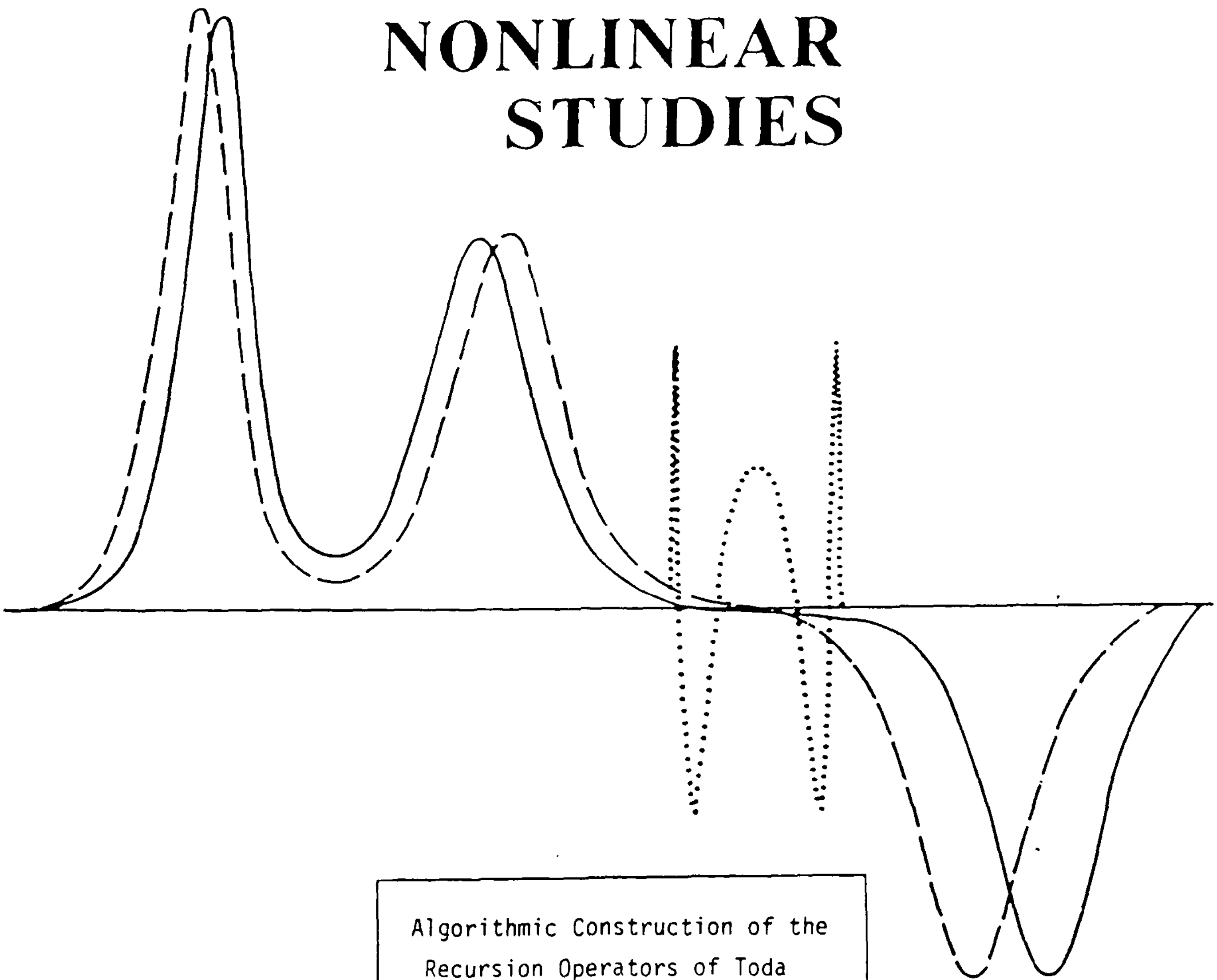

Algorithmic Construction of the Recursion Operators of Toda and Landau-Lifshitz Equation

by

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March 1988

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Abstract

A new approach to the construction of recursion operators of completely integrable system is exhibited. It is explicitly applied to derive the hierarchy of equations of motion of the celebrated Toda lattice as well as the well known Landau-Lifshitz equation.

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†Supported by AFOSR grant #AFOSR-87-0310.

A. Recursion Operator for the Toda Lattice

The equations of motion for the Hamiltonian System of the Toda lattice with Hamiltonian

$$H = \sum_n \left\{ \frac{1}{2} p_n^2 + e^{x_{n+1} - x_n} \right\} \tag{1}$$

are:

$$x_{n,t} = p_n, \qquad p_{n,t} = e^{x_{n+1} - x_n} - e^{x_n - x_{n-1}} \tag{2}$$

and the shift operator of the Lax pair acting on the vector

$$\psi_n = \begin{bmatrix} t_n \\ \epsilon_n \end{bmatrix} \tag{3}$$

is given by (Takhtadzhan and Fadeev (1979))

$$L_n(\lambda)\psi_n = \psi_{n+1} \tag{4}$$

where

$$L_n(\lambda) = \begin{bmatrix} p_n + \lambda & -e^{x_n} \\ e^{-x_n} & 0 \end{bmatrix} \tag{5}$$

After introducing

$$v_n \equiv \epsilon_{n+1} \tag{6}$$

(4) yields the following second order difference equation for v_n:

$$\lambda v_n = e^{x_{n+1} - x_n} v_{n+1} + (-p_n) v_n + v_{n-1} \tag{7}$$

Define now

$$a_n \equiv e^{x_{n+1} - x_n} \quad , \quad b_n \equiv -p_n \tag{8}$$

then (7) is written as:

$$a_n v_{n+1} + b_n v_n + v_{n-1} = \lambda v_n \tag{9}$$

The time evolution of the auxilliary vector ψ_n is expressed in terms of v_n's as

$$v_{n,t} = (A_n v_{n+1} - B_n v_n) a_n \tag{10}$$

and the compatibility of (9), (10) gives:

$$[a_{n,t} + \lambda a_n (A_{n+1} - A_n) - a_n (b_{n+1} A_{n+1} - b_n A_n) - a_n (a_{n+1} B_{n+1} - a_{n-1} B_{n-1})] v_{n+1}$$

$$+[b_{n,t}+\lambda(a_nB_n-a_{n-1}B_{n-1})+b_n(a_{n-1}B_{n-1}-a_nB_n)+a_{n-1}A_{n-1}-a_nA_{n+1}]\,v_n=0 \tag{11}$$

hence both coefficients of v_n and v_{n+1} should vanish i.e.

$$a_{n,t}+\lambda a_n(A_{n+1}-A_n)-a_n(b_{n+1}A_{n+1}-b_nA_n)-a_n(a_{n+1}B_{n+1}-a_{n-1}B_{n-1})=0 \tag{12}$$

and

$$b_{n,t}+\lambda(a_nB_n-a_{n-1}B_{n-1})+b_n(a_{n-1}B_{n-1}-a_nB_n)+a_{n-1}A_{n-1}-a_nA_{n+1}=0 \tag{13}$$

One may postulate

$$A_n=\sum_{j=0}^{N}A_n^{(j)}\lambda^j \quad ,B_n=\sum_{j=0}^{N}B_n^{(j)}\lambda^j \tag{14}$$

So after substitution of (15) into (12), (13) and equating coefficients of λ^j, one obtains the following equations:

$$a_n(A_{n+1}^{(N)}-A_n^{(N)})=0 \quad ,a_nB_n^{(N)}-a_{n-1}B_{n-1}^{(N)}=0 \tag{15}$$

$$a_{n,t}=a_n(b_{n+1}A_{n+1}^{(0)}-b_nA_n^{(0)})+a_n(a_{n+1}B_{n+1}^{(0)}-a_{n-1}B_{n-1}^{(0)}) \tag{16}$$

$$b_{n,t}=b_n(a_nB_n^{(o)}-a_{n-1}B_{n-1}^{(o)})+a_nA_{n+1}^{(o)}-a_{n-1}A_{n-1}^{(o)} \tag{17}$$

$$a_n(A_{n+1}^{(j-1)}-A_n^{(j-1)})=a_n(b_{n+1}A_{n+1}^{(j)}-b_nA_n^{(j)})+a_n(a_{n+1}B_{n+1}^{(j)}-a_{n-1}B_{n-1}^{(j)}) \tag{18}$$

$$a_nB_n^{(j-1)}-a_{n-1}B_{n-1}^{(j-1)}=b_n(a_nB_n^{(j)}-a_{n-1}B_{n-1}^{(j)})+a_nA_{n+1}^{(j)}-a_{n-1}A_{n-1}^{(j)} \tag{19}$$

for $j=1,\ldots,n$.

Upon introducing the operators Δ,Δ^+ (cf. Soliani et al., (1983))

$$\Delta u_n\equiv u_{n+1}-u_n$$
$$\Delta^+u_n\equiv u_{n-1}-u_n \tag{20}$$

one may write (17), (18) in matrix form

$$\begin{bmatrix}a_{n,t}\\ b_{n,t}\end{bmatrix}=\begin{bmatrix}a_n(\Delta-\Delta^+)a_n & a_n\Delta b_n\\ -b_n\Delta^+a_n & a_n\Delta-\Delta^+a_n\end{bmatrix}\begin{bmatrix}B_n^{(0)}\\ A_n^{(0)}\end{bmatrix}$$

$$\equiv \Omega \begin{bmatrix} B_n^{(0)} \\ A_n^{(0)} \end{bmatrix} \tag{21}$$

and (18), (19) expressed as:

$$\Theta \begin{bmatrix} B_n^{(j-1)} \\ A_n^{(j-1)} \end{bmatrix} = \Omega \begin{bmatrix} B_n^{(j)} \\ A_n^{(j)} \end{bmatrix} \tag{22}$$

where

$$\Theta \equiv \begin{bmatrix} 0 & a_n \Delta \\ -\Delta^+ a_n & 0 \end{bmatrix} \tag{23}$$

Note that this operator was present by Soliani et al., (1983). The recursion relation takes the form

$$\begin{bmatrix} B_n^{(j-1)} \\ A_n^{(j-1)} \end{bmatrix} = \Theta^{-1}\Omega \begin{bmatrix} B_n^{(j)} \\ A_n^{(j)} \end{bmatrix} \equiv \Psi \begin{bmatrix} B_n^{(j)} \\ A_n^{(j)} \end{bmatrix} \tag{24}$$

From (25) one obtains recursively:

$$\begin{bmatrix} B_n^{(0)} \\ A_n^{(0)} \end{bmatrix} = \Psi^N \begin{bmatrix} B_n^{(N)} \\ A_n^{(N)} \end{bmatrix} \tag{25}$$

and since a solution of (15) is

$$A_n^{(N)} = c \ , \quad B_n^{(N)} = 0 \tag{26}$$

where c is an arbitrary constant, the hierarchy of the Toda lattice is given by:

$$\begin{bmatrix} a_{n,t} \\ b_{n,t} \end{bmatrix} = \Omega \Psi^N \begin{bmatrix} 0 \\ c \end{bmatrix} \tag{27}$$

The first system of equations ($N = 0$, $c = -1$)

$$\begin{bmatrix} a_{n,t} \\ b_{n,t} \end{bmatrix} = \begin{bmatrix} -a_n \Delta b_n \\ \Delta^+ a_n \end{bmatrix} = \begin{bmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{bmatrix} \tag{28}$$

is equivalent to (2), using (8). The second system ($N = 1$) is:

$$\begin{bmatrix} a_{n,t} \\ b_{n,t} \end{bmatrix} = \Omega \Theta^{-1} \Omega \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{29}$$

and, after noting that

$$\Theta^{-1} = \begin{bmatrix} 0 & a_n^{-1}(\Delta^+)^{-1} \\ \Delta^{-1}a_n^{-1} & 0 \end{bmatrix} \tag{30}$$

where

$$(\Delta^{-1}u)_n \equiv -\sum_{j=n}^{+\infty} u_j \tag{31}$$

(29) becomes:

$$\begin{bmatrix} a_{n,t} \\ b_{n,t} \end{bmatrix} = \begin{bmatrix} a_n(a_{n+1} - 2a_n + a_{n-1}) - a_n(b_{n+1}^2 - b_n^2) \\ -b_n(a_{n-1} - a_n) - a_n(b_{n+1} - b_n) + a_{n-1}b_{n-1} \quad a_n b_n \end{bmatrix} \tag{32}$$

B. Landau-Lifshitz Equation

The Landau-Lifshitz equation (LL) is given by

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J\mathbf{S} \tag{1}$$

where J is the diagonal matrix

$$J = \text{ diag } (J_1, J_2, J_3) \tag{2}$$

and $\mathbf{S}$ is the classical unit spin $\mathbf{S} = (S_1, S_2, S_3)$, i.e.,

$$\mathbf{S} \cdot \mathbf{S} = 1. \tag{3}$$

It is well known that (1) is completely integrable and Sklyanin (1979) and others presented its Lax-pair. Since the LL equation is the continuum limit of the equation of motion of the quantum non-isotropic Heisenberg Hamiltonian (the so-called XYZ), it is not surprising that the Lax pair is expressed in terms of Jacobi elliptic functions. The algebraic structure of (1) was studied in detail by Date, Jimbo, Kashiwara and Miwa (1983) who derived its quasi-periodic solutions as well. Furthermore, Fuchssteiner (1984) studied its master-symmetries.

Consider the equation for the auxilliary vector ψ given by

$$\psi_x = -i\left(\sum_{j=1}^{3} S_j W_j \sigma_j\right)\psi \equiv -iL\psi \tag{4}$$

while L may be viewed as the shift operator associated with the Lax pair. The operators σ_j are the Pauli spin operators given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

and the Jacobi elliptic functions W_j are given by Sklyanin as:

$$W_1 = \rho \frac{1}{sn(u,k)}$$

$$W_2 = \rho \frac{dn(u,k)}{sn(u,k)} \tag{6}$$

$$W_3 = \rho \frac{cn(u,k)}{sn(u,k)}$$

with the modulus k given by

$$k = \left\{ \frac{J_2 - J_1}{J_3 - J_1} \right\}^{1/2} \quad 0 < k < 1 \tag{7}$$

and the arbitrary normalization parameter ρ as well as the parameters α, β are defined by

$$W_1^2 - W_3^2 = \frac{1}{4}(J_3 - J_1) \equiv \alpha \tag{8a}$$

$$W_2^2 - W_3^2 = \frac{1}{4}(J_3 - J_2) \equiv \beta \tag{8b}$$

Formally, one may express the time evolution of the auxilliary vector ψ as

$$\psi_t = -iV\psi \tag{9}$$

and the structure of the operator L suggests that V has similar form, i.e. one may postulate

$$\psi_t = -i \left\{ \sum_{j=1}^{3} W_j V_j \sigma_j \right\} \psi \tag{10}$$

with the compatibility condition

$$L_t - V_x - i[L, V] = 0 \tag{11}$$

that takes the form

$$\sum_{j=1}^{3} S_{j,t} W_j \sigma_j - \sum_{j=1}^{3} V_{j,x} W_j \sigma_j - i \left[\sum_{j=1}^{3} S_j W_j \sigma_j, \sum_{j=1}^{3} V_j W_j \sigma_j \right] = 0 \tag{12}$$

Equating coefficients of σ_j for $j = 1, 2, 3$, one obtains

$$S_{1,t} = \frac{2W_2 W_3}{W_1}(S_3 V_2 - S_2 V_3) + V_{1,x} \tag{13}$$

as well as other cyclic permutations.

It is convenient to introduce the parametrization

$$\lambda \equiv \frac{1}{2} W_1 W_2 W_3, \quad \mu \equiv W_3^2 \tag{14}$$

with the immediate identity

$$\lambda^2 = \frac{1}{4}\mu(\mu+\alpha)(\mu+\beta) \tag{15}$$

where α, β have been defined by (8a), (8b). Thus, (13) and its cyclic permutations take the form

$$S_{1,t} = \frac{\mu(\mu+\beta)}{\lambda}(S_3 V_2 - S_2 V_3) + V_{1,x} \tag{16a}$$

$$S_{2,t} = \frac{(\mu+\alpha)\mu}{\lambda}(S_1 V_3 - S_3 V_1) + V_{2,x} \tag{16b}$$

$$S_{3,t} = \frac{(\mu+\beta)(\mu+\alpha)}{\lambda}(S_2 V_1 - S_1 V_2) + V_{3,x} \tag{16c}$$

One may formally represent the operators V_k by the finite expansions

$$V_1 = \frac{\mu(\mu+\beta)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_1^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_1^{(j)} \tag{17a}$$

$$V_2 = \frac{(\mu+\alpha)\mu}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_2^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_2^{(j)} \tag{17b}$$

$$V_3 = \frac{(\mu+\beta)(\mu+\alpha)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_3^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_3^{(j)} \tag{17c}$$

In other words, determination of the operators $a_k^{(j)}$, $b_l^{(m)}$ is equivalent to a determination of V. Upon substitution of (17) in (16a) one obtains

$$S_{1,t} = \frac{\mu(\mu+\beta)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_{1,x}^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_{1,x}^{(j)}$$

$$-\frac{\mu(\mu+\beta)}{\lambda}\left[S_2\left(\frac{(\mu+\alpha)(\mu+\beta)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_3^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_3^{(j)}\right)\right.$$

$$-S_3\left(\frac{\mu(\mu+\alpha)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_2^{(j)}+\sum_{j=0}^{n}\mu^{n-j}b_2^{(j)}\right)\Bigg] \tag{18}$$

namely

$$S_{1,t}=\frac{\mu(\mu+\beta)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}(a_{1,x}^{(j)}-S_2b_3^{(j)}+S_3b_2^{(j)})+\sum_{j=0}^{n}\mu^{n-j}b_{1,x}^{(j)}$$

$$-4(\mu+\beta)S_2\sum_{j=0}^{n}\mu^{n-j}a_3^{(j)}+4\mu S_3\sum_{j=0}^{n}\mu^{n-j}a_2^{(j)} \tag{19}$$

or

$$S_{1,t}=\frac{\mu(\mu+\beta)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}(a_{1,x}^{(j)}-S_2b_3^{(j)}+S_3b_2^{(j)})+\sum_{j=0}^{n}\mu^{n-j}[b_{1,x}^{(j)}-4\beta S_2a_3^{(j)}]$$

$$-4\sum_{j=-1}^{n-1}\mu^{n-j}[S_2a_3^{(j+1)}-S_3a_2^{(j+1)}] \tag{20}$$

Similarly, the other two equations are given by

$$S_{2,t}=\frac{(\mu+\alpha)\mu}{\lambda}\sum_{j=0}^{n}\mu^{n-j}(a_{2,x}^{(j)}-S_3b_1^{(j)}+S_1b_3^{(j)})+\sum_{j=0}^{n}\mu^{n-j}(b_{2,x}^{(j)}+4\alpha S_3a_1^{(j)})$$

$$-4\sum_{j=-1}^{n-1}\mu^{n-j}(S_3a_1^{(j+1)}-S_1a_3^{(j+1)}) \tag{20b}$$

$$S_{3,t}=\frac{(\mu+\beta)(\mu+\alpha)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}(a_{3,x}^{(j)}-S_1b_2^{(j)}+S_2b_1^{(j)})$$

$$+\sum_{j=0}^{n}\mu^{n-j}(b_{3,x}^{(j)}-4\alpha S_1a_2^{(j)}+4\beta S_2a_1^{(j)})-4\sum_{j=-1}^{n-1}\mu^{n-j}(S_1a_2^{(j+1)}-S_2a_1^{(j+1)}) \tag{20c}$$

Equating coefficients of μ^j and $\lambda^{-1}\mu^j$ independently one obtains

$$\mathbf{S}\times\mathbf{a}^{(0)}=0 \tag{21}$$

$$\mathbf{S}\times\mathbf{b}^{(j)}=\mathbf{a}_x^{(j)}\quad ;j=0,1,\ldots,n \tag{22}$$

$$\mathbf{S} \times \mathbf{a}^{(j+1)} = \frac{1}{4}\mathbf{b}_x^{(j)} - (A\mathbf{S}) \times \mathbf{a}^{(j)} \quad ; j = 0, 1, \ldots, n-1 \tag{23}$$

$$\mathbf{S}_t = \mathbf{b}_x^{(n)} - 4(A\mathbf{S}) \times \mathbf{a}^{(n)} \tag{24}$$

where A is diagonal matrix given by

$$A = \text{ diag } (\alpha, \beta, 0) \tag{25}$$

First, one solves (22) for $b^{(j)}$

$$\mathbf{b}^{(j)} = -\mathbf{S} \times \mathbf{a}_x^{(j)} + g_j\mathbf{S} \tag{26}$$

where g_j is a scalar function of x to be determined by requiring the solvability condition for (23):

$$\left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S} = 0 \tag{27}$$

This condition gives:

$$g_{j,x} = \left\{\mathbf{S}_x \times \mathbf{a}_x^{(j)} + (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S} \tag{28}$$

i.e.

$$g_j = \partial^{-1}\left(\left\{\mathbf{S}_x \times \mathbf{a}_x^{(j)} + (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S}\right) \tag{29}$$

where ∂^{-1} indicates antiderivative with respect to x.

Then

$$\mathbf{b}_x^{(j)} = -\mathbf{S} \times \mathbf{a}_{xx}^{(j)} + \left[(4A\mathbf{S}) \times \mathbf{a}^{(j)} \cdot \mathbf{S}\right]\mathbf{S}$$

$$+ \left[\partial^{-1}\left(\left\{\mathbf{S}_x \times \mathbf{a}_x^{(j)} + (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S}\right)\right]\mathbf{S}_x \tag{30}$$

Now (23) yields:

$$\mathbf{a}^{(j+1)} = -\frac{1}{4}\mathbf{S} \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} + f_{j+1}\mathbf{S} \tag{31}$$

where the scalar f_{j+1} is to be determined by the requirement

$$\mathbf{a}_x^{(j+1)} \cdot \mathbf{S} = 0 \ . \tag{32}$$

for (22) to be solvable for $\mathbf{b}^{(j)}$. Using (31), (32) yields

$$f_{j+1,x} = \frac{1}{4}\mathbf{S}_x \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S} \tag{33}$$

i.e.

$$f_{j+1} = \frac{1}{4}\partial^{-1}\left(\mathbf{S}_x \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S}\right) \tag{34}$$

so,

$$\mathbf{a}^{(j+1)} = -\frac{1}{4}\mathbf{S} \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\}$$

$$+\frac{1}{4}\left[\partial^{-1}\left(\mathbf{S}_x \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S}\right)\right]\mathbf{S} \tag{35}$$

Finally, introducing the operators:

$$\Theta^{-1}\mathbf{a} \equiv \mathbf{S} \times \mathbf{a} + \left[\partial^{-1}(\mathbf{S}_x \times \mathbf{a} \cdot \mathbf{S})\right]\mathbf{S} \tag{36}$$

and

$$\Omega\mathbf{a} \equiv \frac{1}{4}\{-\mathbf{S} \times \mathbf{a}_{xx} - (4A\mathbf{S}) \times \mathbf{a} + [(4A\mathbf{S}) \times \mathbf{a} \cdot \mathbf{S}]\mathbf{S}$$

$$+\left[\partial^{-1}(\{\mathbf{S}_x \times \mathbf{a}_x + (4A\mathbf{S}) \times \mathbf{a}\} \cdot \mathbf{S})\right]\mathbf{S}_x\} \tag{37}$$

we can write (35) and (24) as:

$$\mathbf{a}^{(j+1)} = \Theta^{-1}\Omega\mathbf{a}^{(j)} \equiv \Psi\mathbf{a}^{(j)} \tag{38}$$

and

$$\mathbf{S}_t = 4\Omega\mathbf{a}^{(n)} \tag{39}$$

Next, one has to deal with the "starting points" of the recursion, $\mathbf{a}^{(0)}$, $\mathbf{b}^{(0)}$. It is best illustrated by an explicit derivation of the hierarchy (39) for $n = 1$:

From (21), solving for $\mathbf{a}^{(0)}$, one obtains:

$$\mathbf{a}^{(0)} = F_o\mathbf{S} \tag{40}$$

It turns out that F_o is a constant in order to be able to solve (22) for $\mathbf{b}^{(0)}$:

$$\mathbf{b}^{(0)} = -F_o\mathbf{S} \times \mathbf{S}_x + G_o\mathbf{S} \tag{41}$$

where G_o is a new constant in order that (23) be solvable for $\mathbf{a}^{(1)}$:

$$\mathbf{a}^{(1)} = f_1\mathbf{S} + \frac{1}{4}\{G_o(\mathbf{S} \times \mathbf{S}_x) + F_o[\mathbf{S}_{xx} - (\mathbf{S} \cdot \mathbf{S}_{xx})\mathbf{S}]\}$$

$$+F_o\{(\mathbf{S}\cdot A\mathbf{S})\mathbf{S}-A\mathbf{S}\} \tag{42}$$

Since $\mathbf{a}_x^{(1)}$ has to be normal to $\mathbf{S}$,

$$f_{1,x}-F_o\left\{\frac{1}{4}(\mathbf{S}_x\cdot\mathbf{S}_{xx})-\mathbf{S}_x\cdot A\mathbf{S})\right\}=0 \tag{43}$$

Since $\mathbf{S}$ is a unit vector, i.e. $\mathbf{S}\cdot\mathbf{S}=1$, one has:

$$\mathbf{S}\cdot\mathbf{S}_{xx}=-\mathbf{S}_x\cdot\mathbf{S}_x \tag{44}$$

and

$$\mathbf{S}\cdot\mathbf{S}_{xxx}=\frac{3}{2}(\mathbf{S}\cdot\mathbf{S}_{xx})_x \tag{45}$$

so (43) yields:

$$f_1=F_1+\frac{1}{8}F_o\left[(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}\right] \tag{46}$$

where F_1 is a constant. Hence

$$\mathbf{a}^{(1)}=\left\{F_1+\frac{1}{8}F_o\left[(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}\right]\right\}\mathbf{S}+\frac{1}{4}G_o\mathbf{S}\times\mathbf{S}_x$$

$$+\frac{1}{4}F_o(\mathbf{S}_{xx}-(\mathbf{S}\cdot\mathbf{S}_{xx})\mathbf{S})+F_o\left[(\mathbf{S}\cdot A\mathbf{S})\mathbf{S}-A\mathbf{S}\right] \tag{47}$$

and

$$\mathbf{a}_x^{(1)}=\left\{F_1+\frac{1}{8}F_o\left[(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}\right]\right\}\mathbf{S}_x$$

$$+\frac{F_o}{8}\{(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}_x+(\mathbf{S}_{xxx}-4A\mathbf{S}_x)\cdot\mathbf{S}\}\mathbf{S}$$

$$+\frac{1}{4}G_o\mathbf{S}\times\mathbf{S}_{xx}+\frac{1}{4}F_o\{\mathbf{S}_{xxx}-(\mathbf{S}_x\cdot\mathbf{S}_{xx})\mathbf{S}-(\mathbf{S}\cdot\mathbf{S}_{xxx})\mathbf{S}-(\mathbf{S}\cdot\mathbf{S}_{xx})\mathbf{S}_x\}$$

$$+F_o\{2(\mathbf{S}_x\cdot A\mathbf{S})\mathbf{S}+(\mathbf{S}\cdot A\mathbf{S})\mathbf{S}_x-A\mathbf{S}_x\} \tag{48}$$

Then, solving (22) for $\mathbf{b}^{(1)}$, one gets

$$\mathbf{b}^{(1)}=g_1\mathbf{S}-\mathbf{S}\times\mathbf{a}_x^{(1)} \tag{49}$$

where g_1 has to satisfy the equation (cf. (28))

$$g_{1,x} = \frac{1}{4} G_o \left[\mathbf{S}_x \cdot \mathbf{S}_{xx} + 4A\mathbf{S} \cdot \mathbf{S}_x \right] - \frac{1}{4} F_o \left[\mathbf{S}_x \times (4A\mathbf{S}_x) \cdot \mathbf{S} \right.$$

$$\left. + \mathbf{S}_{xx} \times (4A\mathbf{S}) \cdot \mathbf{S} - \mathbf{S}_x \times \mathbf{S}_{xxx} \cdot \mathbf{S} \right] \tag{50}$$

i.e.

$$g_{1,x} = \frac{1}{8} G_o \left[\mathbf{S}_x \cdot \mathbf{S}_x + 4A\mathbf{S} \cdot \mathbf{S} \right]_x$$

$$+ \frac{1}{4} F_o \left[\mathbf{S}_x \times (\mathbf{S}_{xx} - 4A\mathbf{S}) \cdot \mathbf{S} \right]_x \tag{51}$$

and because of (45):

$$g_1 = G_1 - \frac{1}{8} G_o \left[(\mathbf{S}_{xx} - 4A\mathbf{S}) \cdot \mathbf{S} \right] + \frac{1}{4} F_o \left[\mathbf{S}_x \times (\mathbf{S}_{xx} - 4A\mathbf{S}) \cdot \mathbf{S} \right] \tag{52}$$

One may set $n = 1$ in equation (24). The resulting evolution equation contains the arbitrary constants G_1, F_1, G_o, F_o. By letting all but one vanish, one obtains the hierarchy of evolution equations as:

$$\text{(i)} \quad \mathbf{S}_t = \mathbf{S}_x \tag{53}$$

$$\text{(ii)} \quad \mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + (4A\mathbf{S}) \times \mathbf{S}$$

which is the same as

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J\mathbf{S} \tag{54}$$

because of (8a,b) and (25).

$$\text{(iii)} \quad \mathbf{S}_t = \mathbf{S}_{xxx} + \frac{3}{2} \left[(\mathbf{S}_x \cdot \mathbf{S}_x) - J\mathbf{S} \cdot \mathbf{S} + J_3 \right] \mathbf{S}_x + 3(\mathbf{S}_x \cdot \mathbf{S}_{xx}) \mathbf{S} \tag{55}$$

This equation was obtained by Date, Jimbo, Kashiwara and Miwa (1983)

$$\text{(iv)} \quad \mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xxxx} + \mathbf{S}_x \times \mathbf{S}_{xxx} - \frac{1}{2} \left[3\mathbf{S} \cdot \mathbf{S}_{xx} + \mathbf{S} \cdot J\mathbf{S} \right] \mathbf{S} \times \mathbf{S}_{xx}$$

$$+ \left[3(\mathbf{S}_x \cdot \mathbf{S}_{xx}) - \mathbf{S}_x \cdot J\mathbf{S} \right] \mathbf{S} \times \mathbf{S}_x - \left[\mathbf{S}_x \times (\mathbf{S}_{xx} + J\mathbf{S}) \cdot \mathbf{S} \right] \mathbf{S}_x$$

$$+\frac{1}{2}\left[3(\mathbf{S}\cdot\mathbf{S}_{xx})+(\mathbf{S}\cdot J\mathbf{S})\right](J\mathbf{S})\times\mathbf{S}$$

$$-\left[\mathbf{S}_x\times(\mathbf{S}_{xx}+J\mathbf{S})\cdot\mathbf{S}\right]_x\mathbf{S}+(\mathbf{S}\times J\mathbf{S}_x)_x+\mathbf{S}_{xx}\times(J\mathbf{S}).$$

Detailed account of this work, in particular the bi-Hamiltonian formulation and the connection with the master-symmetry approach, will be published elsewhere.

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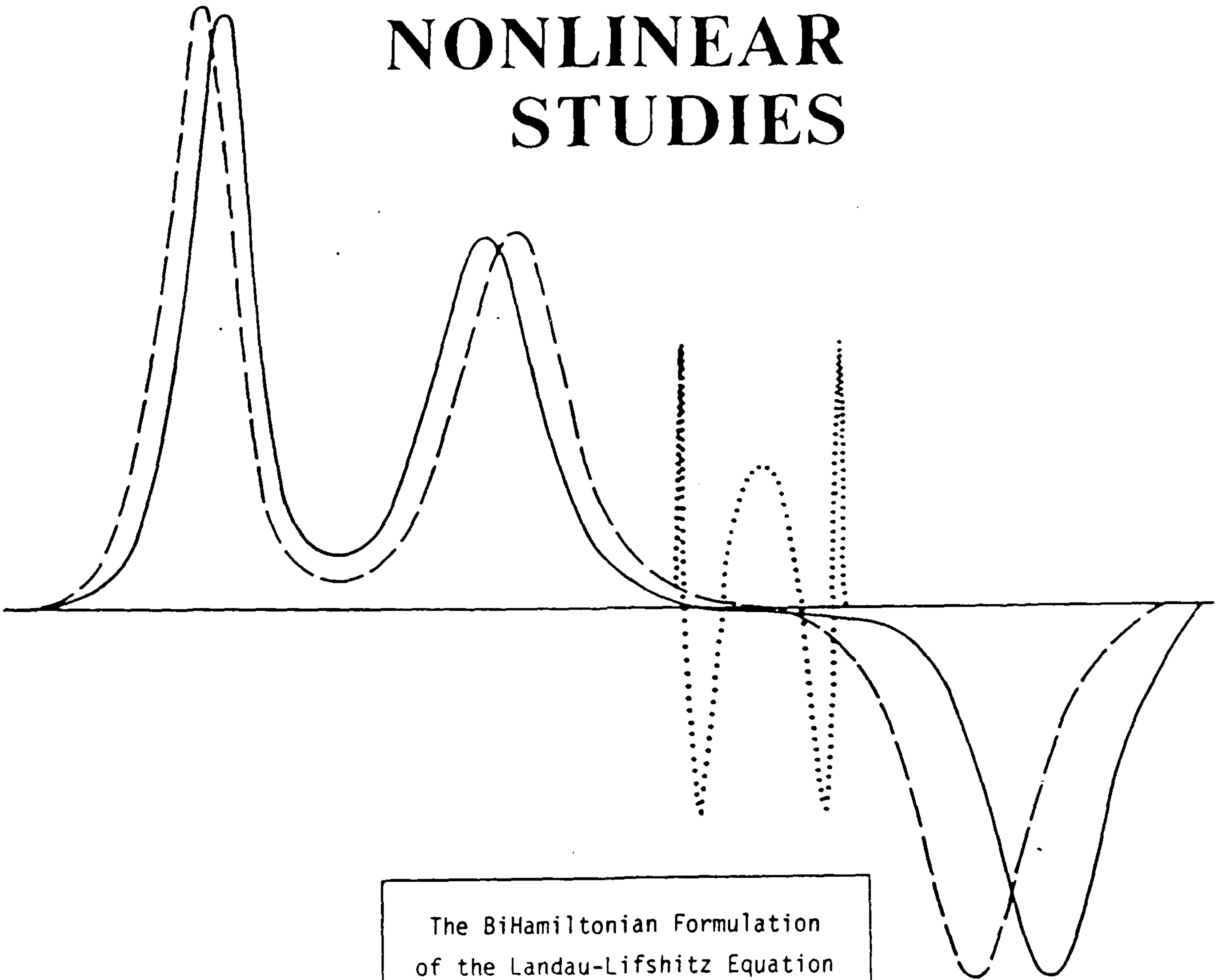

The BiHamiltonian Formulation of the Landau-Lifshitz Equation

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April 1988

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April 27, 1988

Abstract

The Landau-Lifshitz (LL) equation is a universal model for integrable magnetic systems. It contains the sine-Gordon (SG), nonlinear Schrödinger (NLS) and the Heisenberg model (HM) equations as particular or limiting cases. It is well known that the NLS, SG and HM equations possess *recursion operators*. A recursion operator of an equation in Hamiltonian form generates (a) a hierarchy of integrable equations, (b) a second Hamiltonian operator and more generally a hierarchy of Poisson structures. Here we obtain algorithmically the recursion operator of the LL equation and, hence, we establish its bi-Hamiltonian formulation.

INS #89

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*Supported in part by AFOSR Grant #AFOSR-87-0310

1 Introduction

The LL equation describes nonlinear spin waves in an anisotropic ferromagnet. It is given by

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx} + \mathbf{S} \wedge J\mathbf{S} \tag{1.1a}$$

where

$$J = diag(J_1, J_2, J_3), \quad \mathbf{S} = (S_1, S_2, S_3), \quad |\mathbf{S}|^2 = \mathbf{S} \bullet \mathbf{S} = 1. \tag{1.1b}$$

In the above the diagonal matrix J is a measure of the anisotropy, $J_1 < J_2 < J_3$, $\mathbf{S}$ is an x- and t-dependent vector of unit norm in $\mathbf{R}^3$, and $\bullet$, $\wedge$ denote the usual scalar and vector products.

The partially anisotropic HM and the HM equations correspond to $J_1 = J_2 < J_3$, and $J_1 = J_2 = J_3$ respectively. It was pointed out in [1] that the LL is the most general magnet model admitting an r-matrix formulation. Furthermore, both the SG and NLS equations are limiting cases of the LL equation. The analysis of the LL is technically more complicated than that of HM, SG and NLS. This is because the isospectral linear eigenvalue problem associated with LL involves elliptic functions [2]:

$$U_x(x,t,\lambda) = -i\left(\Sigma_{j=1}^3 S_j(x,t)W_j(\lambda)\sigma_j\right) U(x,t,\lambda) \doteq -iLU, \tag{1.2a}$$

where the Pauli spin matrices are given by

$$\sigma_1 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2b}$$

and

$$W_1(\lambda) \doteq \rho\frac{1}{sn(\lambda,k)}, \quad W_2(\lambda) \doteq \rho\frac{dn(\lambda,k)}{sn(\lambda,k)}, \quad W_3(\lambda) \doteq \rho\frac{cn(\lambda,k)}{sn(\lambda,k)}, \tag{1.3a}$$

with

$$k \doteq \left(\frac{J_2 - J_1}{J_3 - J_1}\right)^{1/2}, \quad 0 < k < 1, \quad \rho \doteq \sqrt{J_3 - J_1}. \tag{1.3b}$$

In the isospectral problems associated with the HM, SG and NLS equations the spectral parameter λ ranges over the complex plane $\mathbf{C}$, however the natural range of λ in (1.2) is an elliptic curve: The torus $E = \mathbf{C}/\Gamma$ where Γ is the lattice generated by $4K$ and $4iK'$, where K and K' are the complete elliptic integrals of moduli k and $k' = \sqrt{1-k^2}$.

The Lax pair of the LL was found by Sklyanin [2] (see also [3]), who also obtained the action-angles variables (for rapidly approaching a fixed unit vector boundary conditions) by introducing the notion of the classical r-matrix. The initial value problem for similar data was studied by Mikhailov [4] (see also [5]) using a Riemann-Hilbert problem on an eliptic curve. A general description of finite-gap solutions was given in [6] and explicit formulae were obtained in [7] and [8] in terms of Prym theta functions.

Algebraic properties of the LL were studied in [7] where also the next member of its hierarchy was explicitly given. Fuchssteiner [9] presented hierarchies of time-independent symmetries, time-dependent symmetries and conserved quantities using the notion of a mastersymmetry introduced in [10]. However, the recursion operator could not be found and hence its bi-Hamiltonian formulation could not be established. This is a serious disadvantage since the bi-Hamiltonian property appears to be a fundamental property underlying integrability [11-15]. Indeed, the bi-Hamiltonian formulation of NLS and SG are well established. Also the recursion operator and the hierarchy of Hamiltonian operators associated with the HM have been found in [16] using the gauge equivalence of the HM to the NLS ([17], [18]).

There exist various approaches in the literature for constructing recursion oeprators [19]. We favor the one which uses the associated isospectral problem. Indeed, this approach has also been successful for obtaining recursion operators in lattices [20] and in multidimensions [21]. Also, it has the advantage to yield hereditary recursion operators [22]. In §2 we illustrate our method by deriving the recursion operator of the HM equation; this operator coincides with the one given in [16]. In §3 we derive the recursion operator of the LL equation and establish its bi-Hamiltonian factorization [23].

The method of deriving the recursion operator from an isospectral problem makes crucial use of a certain expansion in powers of the spectral parameter λ. The main difficulty we encountered in applying this method to LL stemmed from the fact that λ moves on an elliptic curve. This problem was bypassed by using the parametrization

$$\nu \doteq \frac{1}{2}W_1 W_2 W_3, \quad \mu \doteq W_3^2, \tag{1.4}$$

$$\nu^2 = \frac{1}{4}\mu(\mu+\alpha)(\mu+\beta); \quad \alpha \doteq -\frac{1}{4}(J_1 - J_3), \quad \beta \doteq -\frac{1}{4}(J_2 - J_3). \tag{1.5}$$

This paper is organized as follows. In §1.1 we review the basic notions of symmetries, gradients of conserved quantities, recursion operators and Hamiltonian operators. In §1.2 we establish the connection between these results and thos of Fuchssteiner [9] by showing how the recursion operator derived in this paper algorithmically implies tlre mastersymmetry found in [9]. In §2, §3 we derive the factorizable recursion operators of the HM and LL equations respectively.

1.1 Basic Notions

We consider the evolution equation (1.1) in the abstract form

$$\mathbf{S}_t = K(\mathbf{S}) \tag{1.6}$$

Let E denote the vector space of C^∞ – *maps* from $\mathbf{R}$ into $\mathbf{R}^3$ and let TE denote the space of suitable C^∞ vectorfields on E. The manifold on which the flow (1.6) takes place is denoted by M and the space of its smooth vector fields by TM. Clearly , M is a subspace of E such that $\mathbf{S}\epsilon E$ satisfies $\mathbf{S}\bullet\mathbf{S} = 1$. Similarly TM is a subspace of TE such that $V(\mathbf{S})\epsilon T_{\mathbf{S}}E$ satisfies $V(\mathbf{S})\bullet\mathbf{S} = 0$, i.e. $V(\mathbf{S}(x))$ belongs to the tangent plane of the unit sphere at $\mathbf{S}(x)$.

In TM we define the usual Lie-bracket by

$$[K, G]_L \doteq K'[G] - G'[K]. \tag{1.7a}$$

where $K'[G]$ denotes the Frechét derivative of K in the direction G, i.e.

$$K'[G] \doteq \frac{\partial}{\partial \epsilon} K(S + \epsilon G)|_{\epsilon=0}. \tag{1.7b}$$

Let T^*M be the dual of TM with respect to the bilinear form

$$(\gamma, \sigma) \doteq \int_{\mathbf{R}} dx \gamma \bullet \sigma; \quad \gamma \epsilon T^* M, \quad \sigma \epsilon TM. \tag{1.8}$$

Let $I : M \rightarrow \mathbf{R}$ be a functional; then its gradient, ∇I, is defined by

$$I'[v] \doteq (\nabla I, v), \quad v \epsilon TM. \tag{1.9}$$

It is well known that a function f is a gradient $iff \ \ f' = (f')^+$, where the adjoint L^+ of an operator L is defined by $(L^+ \gamma, \sigma) = (\gamma, L\sigma)$. In order to make the gradient unique we consider its projection onto the tangent plane of the unit sphere in $\mathbf{R}^3$ at the point $S(x)$; i.e. $\gamma \bullet S = 0$.

The conserved quantities of the LL equation take the form

$$I = \int_{\mathbf{R}} dx(\Gamma(S) - \Gamma(e)), \quad e \doteq (0, 0, 1)^+, \tag{1.10}$$

where we have assumed that $S \rightarrow e$ as $x \rightarrow \pm\infty$. As an example consider

$$H_o = \int_{\mathbf{R}} dx(\Gamma_0(S) - \Gamma_0(e)), \quad \Gamma_o \doteq \frac{1}{2}(S \bullet JS - S_x \bullet S_x), \tag{1.11a}$$

then

$$H_o'[v] = \int_{\mathbf{R}} dx(v \bullet JS - v_x \bullet S_x) = \int_{\mathbf{R}} dx v \bullet (JS + S_{xx}).$$

thus

$$\nabla H_o = \pi(S_{xx} + JS), \quad \pi a \doteq -S \wedge (S \wedge a) = a - (a \bullet S)S. \tag{1.11b}$$

(i) The hierarchy of the LL equation consists of all flows which commute with (1.1); i.e. it consists of all time-independent symmetries σ. We recall that σ is a symmetry of (1.1) iff

$$\frac{\partial \sigma}{\partial t} + [\sigma, K]_L = 0, \quad \sigma \epsilon TM. \tag{1.12}$$

(ii) An equation (1.6) is a Hamiltonian system iff it can be written in the form

$$S_t = \Theta \nabla H, \tag{1.13a}$$

where Θ is a Hamiltonian operator, i.e. Θ is skew-symmetric with respect to (1.8) and it satisfies, also, the Jacobi identity:

$$(\nabla I_1, \Theta'[\nabla I_2]\nabla I_3) + \text{ cyclic permutations } = 0, \quad \nabla I_i \epsilon T^* M, \quad i = 1, 2, 3. \tag{1.13b}$$

and H is a functional. The Hamiltonian operator Θ induces the following Poisson bracket,

$$\{I_1, I_2\} \doteq (\nabla I_1, \Theta \nabla I_2). \tag{1.14}$$

(iii) A functional I is a conserved quantity of (1.6) iff $I'[K] = 0$, or (cf. 1.13a).

$$I'[K] = (\nabla I, \Theta \nabla H) = \{I, H\} = 0.$$

It turns out that it is more convenient to work with gradients of conserved quantities; these conserved gradients satisfy

$$\frac{\partial \gamma}{\partial t} + \gamma'[K] + (K')^{+}[\gamma] = 0, \quad \gamma \doteq \nabla I. \tag{1.15}$$

For Hamiltonian systems there is an isomorphism between the Lie commutator (1.7a) and the Poisson bracket (1.14), [10]-[12]:

$$[\Theta \nabla I_1, \Theta \nabla I_2]_L = \Theta \nabla(\{I_1, I_2\}). \tag{1.16}$$

This isomorphism implies that, for a Hamiltonian system, symmetries and gradients of conserved quantities are related by

$$\sigma = \Theta \nabla I, \quad \sigma \epsilon TM, \quad \nabla I \epsilon T^{*}M. \tag{1.17}$$

It is well known that the LL equation is a Hamiltonian system. Indeed, it can be written in the form:

$$\mathbf{S}_t = \mathbf{S} \wedge \nabla H_o, \tag{1.18}$$

where ∇H_o is defined by (1.11) and $\Theta = \mathbf{S}\wedge$ is a Hamiltonian operator (Θ is obviously skew-symmetric and it is a straightforward exercise to show that it satisfies the Jacobi identity).

Fundamental role in the characterization of the algebraic properties of integrable evolution equations is played by hereditary (Nijenhuis) recursion operators.

If Φ is a hereditary (Nijenhuis) operator then

$$[\Phi^n K, \Phi^m K]_L = 0, \quad ((\Phi^{+})^n \nabla H, \quad \Theta(\Phi^{+})^m \nabla H) = 0, \tag{1.19}$$

and $\Phi^n \Theta$ are Hamiltonian operators compatible with Θ for all $n, m \epsilon \mathbf{N}$. (Two Hamiltonian operators are compatible if their sum is a Hamiltonian operator).
In §§2.3 we derive hereditary recursion operators for HM and LL equations. Then $\Phi^n K$, $(\Phi^{+})^n \nabla H_o$, $\Phi^n(\mathbf{S} \wedge \cdot)$ define hierarchies of commuting symmetries, conserved gradients in involution and Hamiltonian operators respectively.

1.2 Mastersymmetries

The general theory associated with mastersymmetries of evolution equations in one spatial and one temporal dimension is well established [21], [24], [25]. Here we only note that given a time-dependent symmetry σ of the form

$$\sigma = \sigma_0 + t\sigma_1, \tag{1.20a}$$

and a recursion operator Φ, then

$$\tau = \Phi\sigma_0 \tag{1.20b}$$

is a mastersymmetry. Alternatively, if

$$\gamma = \gamma_0 + t\gamma_1 \tag{1.21a}$$

is a time-dependent conserved gradient, and $\Psi = \Phi^+$, then

$$T = \Theta\Psi\gamma_0 \tag{1.21b}$$

is a mastersymmetry.

It turns out that

$$\tau = \mathrm{S} \wedge \Psi_{LL}(x\mathrm{S}), \tag{1.22a}$$

where Ψ_{LL} is the adjoint of the recursion operator of the LL (see equation (3.1)), is a mastersymmetry of the LL equation. This coincides with the one given by Fuchssteiner [9].

2 The Heisenberg Model (HM)

The HM equation is given by

$$\mathrm{S}_t = \mathrm{S} \wedge \mathrm{S}_{xx}, \quad \mathrm{S} \bullet \mathrm{S} = 1. \tag{2.1}$$

Its associated isospectral eigenvalue problem is given by

$$U_x = \frac{i}{\lambda}\Sigma_{j=1}^3 S_j\sigma_j U, \tag{2.2}$$

where λ is the spectral parameter and the Pauli matrices σ_j are defined in (1.2b).

Proposition 2.1

(a) The isospectral eigenvalue problem (2.2) yields the recursion operator Φ_{HM} defined by

$$\Phi_{HM} \doteq -\frac{1}{2}\left[\mathrm{S} \wedge D - \{D^{-1}(\mathrm{S} \wedge \mathrm{S}_x \bullet \cdot)\}\mathrm{S}_x\right]. \tag{2.3}$$

(b) The adjoint of Φ_{HM} with respect to the bilinear form (1.8),

$$\Psi_{HM} \doteq \Phi^+_{HM} = -\frac{1}{2}(\mathrm{S} \wedge D - \{D^{-1}(\mathrm{S} \bullet D\cdot)\}\mathrm{S} \wedge \mathrm{S}_x) \tag{2.4}$$

satisfies

$$\mathrm{S} \wedge (\Psi_{HM}\cdot) = \Phi_{HM}(\mathrm{S} \wedge \cdot). \tag{2.5}$$

(c) The isospectral problem (2.2) is associated with the hierarchy of integrable evolution equations

$$\mathbf{S}_t = \mathbf{S} \wedge \Psi_{HM}^{n-1}(\mathbf{S} \wedge \mathbf{S}_x) = \Phi_{HM}^{n-1}(-\mathbf{S}_x), \quad n = 1, 2, 3, \ldots \quad . \tag{2.6}$$

The HM equation corresponds to $n = 2$.

(d) The hierarchy $\mathbf{S} \wedge \Psi_{HM}^n$, $n = 0, 1, 2, \ldots$ is a hierarchy of Hamiltonian operators. In particular the second Hamiltonian operator of the HM is given by $\Omega_{HM} \doteqdot \mathbf{S} \wedge \Psi_{HM}$, thus the HM is a bi-Hamiltonian system with compatible Hamiltonian operators $\mathbf{S}\wedge$ and Ω_{HM}.

Proof. Given (2.2) we look for compatible flows in the form

$$U_t = -i\Sigma_{\ell=1}^3 V_\ell \sigma_\ell U. \tag{2.7}$$

The compatibility condition $U_{tx} = U_{xt}$ of equations (2.2), (2.7) implies

$$\mathbf{S}_t = \lambda \mathbf{V}_x - 2\mathbf{S} \wedge \mathbf{V}, \quad \mathbf{V} = (V_1, V_2, V_3). \tag{2.8}$$

We seek solutions $\mathbf{V}$ in the form

$$\mathbf{V} = \Sigma_{k=1}^n \mathbf{V}^{(k)} \lambda^{-k}. \tag{2.9}$$

Then (2.8) yields

$$\mathbf{S}_t = \mathbf{V}_x^{(1)} \tag{2.10}$$

$$\mathbf{V}_x^{(j+1)} = 2\mathbf{S} \wedge \mathbf{V}^{(j)}, \quad j = 1, \ldots, n-1, \tag{2.11}$$

$$\mathbf{S} \wedge \mathbf{V}^{(n)} = 0. \tag{2.12}$$

Since $\mathbf{V}_x^{(j)} \bullet \mathbf{S} = 0$, we define $\mathbf{v}^{(j)}$ as follows:

$$\mathbf{v}^{(j)} \doteqdot -\mathbf{S} \wedge \mathbf{V}_x^{(j)}, \tag{2.13}$$

with

$$\mathbf{v}^{(j)} \bullet \mathbf{S} = 0. \tag{2.14}$$

Then equations (2.10)-(2.12) are transformed into

$$\mathbf{S} \wedge \mathbf{S}_t = -\mathbf{v}^{(1)}, \tag{2.15}$$

$$\mathbf{v}^{(j+1)} = -2[\mathbf{S} \wedge (\mathbf{S} \wedge (D^{-1}\{\mathbf{S} \wedge \mathbf{v}^{(j)}\}))], \tag{2.16}$$

$$\mathbf{S} \wedge D^{-1}(\mathbf{S} \wedge \mathbf{v}^{(n)}) = 0. \tag{2.17}$$

We solve equation (2.16) for $\mathbf{v}^{(j)}$ as follows:
Equation (2.16) is equivalent to

$$\mathbf{v}^{(j+1)} = 2D^{-1}\{\mathbf{S} \wedge \mathbf{v}^{(j)}\} - 2(\mathbf{S} \bullet D^{-1}\{\mathbf{S} \wedge \mathbf{v}^{(j)}\})\mathbf{S}.$$

Hence

$$\mathbf{v}_x^{(j+1)} = 2\mathbf{S} \wedge \mathbf{v}^{(j)} - 2(\mathbf{S} \cdot D^{-1}(\mathbf{S} \wedge \mathbf{v}^{(j)})\mathbf{S}_x - 2(\mathbf{S} \bullet D^{-1}\{\mathbf{S} \wedge \mathbf{v}^{(j)}\})_x\mathbf{S}. \tag{2.18}$$

From equation (2.18), taking $\mathbf{S}\wedge$ and $\mathbf{S}\bullet$ of both sides we obtain

$$\mathbf{S} \wedge \mathbf{v}_x^{(j+1)} = -2\mathbf{v}^{(j)} - 2(\mathbf{S} \bullet D^{-1}\{\mathbf{S} \wedge \mathbf{v}^{(j)}\})\mathbf{S} \wedge \mathbf{S}_x. \tag{2.19}$$

and

$$\mathbf{S} \bullet \mathbf{v}_x^{(j+1)} = -2\left(\mathbf{S} \bullet D^{-1}\{\mathbf{S} \wedge \mathbf{v}^{(j)}\}\right)_x,$$

i.e.

$$2\mathbf{S} \bullet D^{-1}\{\mathbf{S} \wedge \mathbf{v}^{(j)}\} = -D^{-1}(\mathbf{S} \bullet \mathbf{v}_x^{(j+1)}). \tag{2.20}$$

Substituting in (2.19), we get

$$\mathbf{v}^{(j)} = -\frac{1}{2}\left(\mathbf{S} \wedge \mathbf{v}_x^{(j+1)} - \{D^{-1}(\mathbf{S} \bullet \mathbf{v}_x^{(j+1)})\}\mathbf{S} \wedge \mathbf{S}_x\right). \tag{2.21}$$

i.e. (cf. (2.4))

$$\mathbf{v}^{(j)} = \Psi \mathbf{v}^{(j+1)}.$$

So,

$$\mathbf{v}^{(1)} = \Psi^{n-1}\mathbf{v}^{(n)},$$

and solving (2.15) and (2.17) we get

$$\mathbf{S}_t = \mathbf{S} \wedge \Psi^{n-1}(\mathbf{S} \wedge \mathbf{S}_x). \tag{2.22}$$

In the Appendix, we show that $\mathbf{S}\wedge$ and Ω_{HM} are compatible Hamiltonian operators thus, establishing the bi-Hamiltonian structure of the HM.

Remarks 2.1.

(i) Equation (1.15) is derived by differentiating $(\gamma, K) = 0$ in the arbitrary direction $\mathbf{v}$, where $\mathbf{v} \bullet \mathbf{S} = 0$. Thus, one can extend the definition of a conserved gradient by allowing functions γ which are not of the form $\pi\tilde{\gamma}$, provided that

$$\mathbf{S} \wedge (\frac{\partial \tilde{\gamma}}{\partial t} + \tilde{\gamma}'[K] + (K')^{+}[\tilde{\gamma}]) = 0, \tag{2.23}$$

$$([\tilde{\gamma}' - (\tilde{\gamma}')^+]\mathbf{a}, \mathbf{b}) = 0, \quad \mathbf{a}, \mathbf{b} \text{ orthogonal to } \mathbf{S}. \tag{2.24}$$

Indeed the starting γ of the HM hierarchy satisfies,

$$\tilde{\gamma} \doteq \mathbf{S} \wedge \mathbf{S}_x, \quad \tilde{\gamma}'[K] + (K')^+[\tilde{\gamma}] = -\frac{3}{2}(\mathbf{S}_x \bullet \mathbf{S}_x)_x \mathbf{S}. \tag{2.25}$$

$$([\tilde{\gamma}' - (\tilde{\gamma}')^+]\mathbf{a}, \mathbf{b}) = (\mathbf{S}_x \wedge \mathbf{a}, \mathbf{b}) = 0. \tag{2.26}$$

(ii) $\Psi_{HM}(\mathbf{S} \wedge \mathbf{S}_x) = \pi \mathbf{S}_{xx} = \nabla H_0$, where

$$H_0 \doteq \int_{-\infty}^{\infty} dx(\Gamma_0(\mathbf{S}) - \Gamma_0(e)), \quad \Gamma_0 = -\frac{1}{2}\mathbf{S}_x \bullet \mathbf{S}_x \tag{2.27}$$

(iii) $\gamma^{(1)} \doteq x\mathbf{S} \wedge \mathbf{S}_x - 2t\mathbf{S}_{xx}$ is a conserved gradient of the HM. Hence

$$\tau \doteq \Theta\Psi_{HM}(x\mathbf{S} \wedge \mathbf{S}_x) = x\mathbf{S} \wedge \nabla H_0 + \mathbf{S} \wedge \mathbf{S}_x \tag{2.28}$$

is a mastersymmetry of HM. This coincides with equation (12) of [9] if $J = 0$.

(iv) It is shown in the Appendix that the operator $\Omega_{HM} \doteq \mathbf{S} \wedge \Psi_{HM}$ satisfies the Jacobi identity. Ω_{HM} is equivalent to $\tilde{\Omega} = \frac{1}{2}(D + D\{\mathbf{S}D^{-1}(\mathbf{S}_x \bullet \cdot)\})$, since $\mathbf{S} \bullet \mathbf{a} = 0$. However, in order to prove the Jacobi identity for $\tilde{\Omega}$ we have to take into account that $\tilde{\Omega}\mathbf{a} \bullet \mathbf{b} = \tilde{\Omega}\mathbf{b} \bullet \mathbf{c} = \tilde{\Omega}\mathbf{c} \bullet \mathbf{a} = 0$ which are Fréchet-derivative consequences of the equations $\mathbf{S} \bullet \mathbf{a} = \mathbf{S} \bullet \mathbf{b} = \mathbf{S} \bullet \mathbf{c} = 0$.

3 The Landau-Lifshitz (LL) Equation.

Proposition 3.1.

(a) The isospectral eigenvalue problem (1.2), yields the recursion operator Ψ_{LL} defined by:

$$\Phi_{LL} \doteq \Phi^2_{HM} - \frac{1}{4}\pi\left((4A\mathbf{S}) \wedge (\mathbf{S} \wedge \cdot) - (D^{-1}\{S \bullet 4A\mathbf{S} \wedge (\mathbf{S} \wedge \cdot)\})\mathbf{S}_x - (D^{-1}\{\mathbf{S} \bullet (\mathbf{S} \wedge \cdot)_x\})(4A\mathbf{S} \wedge \mathbf{S})\right) \tag{3.1a}$$

(b) The adjoint of Φ_{LL} with respect to the bilinear form (1.8) is

$$\Psi_{LL} \doteq \Psi^2_{HM} + \frac{1}{4}\mathbf{S} \wedge \left((4A\mathbf{S}) \wedge \cdot - (D^{-1}\{\mathbf{S} \bullet 4A\mathbf{S} \wedge \cdot)\mathbf{S}_x - (D^{-1}\{\mathbf{S} \bullet D\cdot\})4A\mathbf{S} \wedge \mathbf{S}\right). \tag{3.1b}$$

and satisfies

$$\mathbf{S} \wedge (\Psi_{LL}\cdot) = \Phi_{LL}(\mathbf{S} \wedge \cdot) = \Omega_{LL}. \tag{3.1c}$$

(c) The associated hierarchy of integrable evolution equations is given by

$$\mathbf{S}_t = \mathbf{S} \wedge \Psi_{LL}^{n-1}(\alpha \mathbf{S} \wedge \mathbf{S}_x), \quad n = 1,2,3,..., \quad \alpha = \text{ constant} \tag{3.2a}$$

$$\mathbf{S}_t = \mathbf{S} \wedge \Psi_{LL}^{n-1}(0), \quad n = 1,2,3,... \ . \tag{3.2b}$$

The LL-equation corresponds to (3.2b), n=2. Note that in (3.2b) $D^{-1}(0)$ is understood as a constant.

(d) The hierarchy $\mathbf{S} \wedge \Psi_{LL}^n$, $n = 0,1,2,...$ is a hierarchy of Hamiltonian operators. In particular the second Hamiltonian operator of the LL equation is given by $\Omega_{LL} \doteqdot \mathbf{S}\wedge\Psi_{LL}$. thus the LL is a bi-Hamiltonian system with compatible Hamiltonian operators $\mathbf{S}\wedge$ and Ω_{LL}.

Proof . Given (1.2), we seek compatible flows in the form

$$U_t = -i\left\{\Sigma_{j=1}^3 W_j V_j \sigma_j\right\} U. \tag{3.3}$$

The compatibility condition $U_{tx} = U_{xt}$ of equations (1.2), (3.3) implies

$$\Sigma_{j=1}^3 S_{j,t} W_j \sigma_j - \Sigma_{j=1}^3 V_{j,x} W_j \sigma_j - i\left[\Sigma_{j=1}^3 S_j W_j \sigma_j, \quad \Sigma_{\ell=1}^3 V_\ell W_\ell \sigma_\ell\right] = 0. \tag{3.4}$$

Equating coefficients of σ_j, for $j = 1,2,3$, one obtains

$$S_{1,t} = \frac{2W_2W_3}{W_1}(S_3V_2 - S_2V_3) + V_{1,x}, \tag{3.5}$$

and cyclic permutations.
In terms of the parameters μ, ν (cf. (1.4), (1.5)), we get

$$S_{1,t} = \frac{\mu(\mu+\beta)}{\nu}(S_3V_2 - S_2V_3) + V_{1,x}, \tag{3.6a}$$

$$S_{2,t} = \frac{(\mu+\alpha)\mu}{\nu}(S_1V_3 - S_3V_1) + V_{2,x}, \tag{3.6b}$$

$$S_{3,t} = \frac{(\mu+\beta)(\mu+\alpha)}{\nu}(S_2V_1 - S_1V_2) + V_{3,x}. \tag{3.6c}$$

We seek solutions V_j; $j = 1,2,3$, in the form

$$V_1 = \frac{\mu(\mu+\beta)}{\nu}\Sigma_{j=0}^n \mu^{n-j} a_1^{(j)} + \Sigma_{j=0}^n \mu^{n-j} b_1^{(j)} \tag{3.7a}$$

$$V_2 = \frac{(\mu+\alpha)\mu}{\nu}\Sigma_{j=0}^n \mu^{n-j} a_2^{(j)} + \Sigma_{j=0}^n \mu^{n-j} b_2^{(j)} \tag{3.7b}$$

$$V_3 = \frac{(\mu+\beta)(\mu+\alpha)}{\nu}\Sigma_{j=0}^n \mu^{n-j} a_3^{(j)} + \Sigma_{j=0}^n \mu^{n-j} b_3^{(j)}. \tag{3.7c}$$

Upon substitution of (3.7) in (3.6) one obtains

$$S_{1,t} = \frac{\mu(\mu+\beta)}{\nu}\Sigma_{j=0}^{n}\mu^{n-j}a_{1,x}^{(j)} + \Sigma_{j=0}^{n}\mu^{n-j}b_{1,x}^{(j)}$$

$$-\frac{\mu(\mu+\beta)}{\nu}\left[S_2\left(\frac{(\mu+\alpha)(\mu+\beta)}{\nu}\Sigma_{j=0}^{n}\mu^{n-j}a_3^{(j)} + \Sigma_{j=0}^{n}\mu^{n-j}b_3^{(j)}\right)\right.$$

$$\left.-S_3\left(\frac{\mu(\mu+\alpha)}{\nu}\Sigma_{j=0}^{n}\mu^{n-j}a_2^{(j)} + \Sigma_{j=0}^{n}\mu^{n-j}b_2^{(j)}\right)\right] \tag{3.8}$$

i.e.

$$S_{1,t} = \frac{\mu(\mu+\beta)}{\nu}\Sigma_{j=0}^{n}\mu^{n-j}\left(a_{1,x}^{(j)} - S_2b_3^{(j)} + S_3b_2^{(j)}\right) + \Sigma_{j=0}^{n}\mu^{n-j}\left(b_{1,x}^{(j)} - 4\beta S_2a_3^{(j)}\right)$$

$$-4\Sigma_{j=-1}^{n-1}\mu^{n-j}\left(S_2a_3^{(j+1)} - S_3a_2^{(j+1)}\right), \tag{3.9}$$

and similarly for the other two equations.

Equating coefficients of μ^j and $\nu^{-1}\mu^j$ independently, one obtains

$$\mathbf{S}\wedge\mathbf{a}^{(0)} = 0, \tag{3.10}$$

$$\mathbf{S}\wedge\mathbf{b}^{(j)} = \mathbf{a}_x^{(j)}, \tag{3.11}$$

$$\mathbf{S}\wedge\mathbf{a}^{(j+1)} = \frac{1}{4}\left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S})\wedge\mathbf{a}^{(j)}\right\}, \tag{3.12}$$

$$\mathbf{S}_t = \mathbf{b}_x^{(n)} - (4A\mathbf{S})\wedge\mathbf{a}^{(n)}. \tag{3.13}$$

We define

$$q^{(j)} = -\mathbf{S}\wedge\left\{b_x^{(j)} - (4A\mathbf{S})\wedge\mathbf{a}^{(j)}\right\}. \tag{3.14}$$

Then (3.12) yields

$$\frac{1}{4}\mathbf{q}^{(j)} = \mathbf{a}^{(j+1)} - (\mathbf{S}\bullet\mathbf{a}^{(j+1)})\mathbf{S}. \tag{3.15}$$

Since $\mathbf{a}_x^{(j+1)}\bullet\mathbf{S} = 0$,(cf.3.11),

$$\mathbf{a}^{(j+1)} = \frac{1}{4}(\mathbf{q}^{(j)} - \{D^{-1}(\mathbf{S}\,\dot{\bullet}\,\mathbf{q}_x^{(j)})\}\mathbf{S}). \tag{3.16}$$

Applying the operators $(4A\mathbf{S})\wedge$ and $D(\mathbf{S}\wedge)D$ on (3.16) we obtain

$$(4A\mathbf{S})\wedge\mathbf{a}^{(j+1)} = (A\mathbf{S})\wedge\mathbf{q}^{(j)} - \{D^{-1}(\mathbf{S}\bullet\mathbf{q}_x^{(j)})\}(A\mathbf{S})\wedge\mathbf{S}, \tag{3.17}$$

and

$$-\mathbf{b}_x^{(j+1)} + (\mathbf{S} \bullet \mathbf{b}^{(j+1)})_x \mathbf{S} + (\mathbf{S} \bullet \mathbf{b}^{(j+1)})\mathbf{S}_x = \frac{1}{4} D\{\mathbf{S} \wedge \mathbf{q}_x^{(j)} - [D^{-1}(\mathbf{S} \bullet \mathbf{q}_x^{(j)})]\mathbf{S} \wedge \mathbf{S}_x\}. \quad (3.18)$$

because of (3.11).

Taking $\mathbf{S}\bullet$ of (3.18), (3.12) and (3.17) we get

$$-(\mathbf{S} \bullet \mathbf{b}_x^{(j+1)}) + (\mathbf{S} \bullet \mathbf{b}^{(j+1)})_x = \frac{1}{4} \mathbf{S} \bullet D\{\mathbf{S} \wedge \mathbf{q}_x^{(j)}\}. \quad (3.19)$$

$$\mathbf{S} \bullet \mathbf{b}_x^{(j+1)} = \mathbf{S} \bullet (4A\mathbf{S}) \wedge \mathbf{a}^{(j+1)}. \quad (3.20)$$

and

$$\mathbf{S} \bullet (4A\mathbf{S}) \wedge \mathbf{a}^{(j+1)} = +\frac{1}{4} \mathbf{S} \bullet (4A\mathbf{S}) \wedge \mathbf{q}^{(j)}. \quad (3.21)$$

Therefore

$$\mathbf{S} \bullet \mathbf{b}^{(j+1)} = \frac{1}{4} D^{-1}\{\mathbf{S} \bullet [D\{\mathbf{S} \wedge \mathbf{q}_x^{(j)}\} + (4A\mathbf{S}) \wedge \mathbf{q}^{(j)}]\} \quad (3.22)$$

From (3.14), (3.17), (3.18) and (3.22) we get, (cf. (2.4) also),

$$\mathbf{q}^{(j+1)} = \Psi_{HM}^2 \mathbf{q}^{(j)} + \frac{1}{4} \mathbf{S} \wedge \Big((4A\mathbf{S}) \wedge \mathbf{q}^{(j)} - (D^{-1}\{\mathbf{S} \bullet 4A\mathbf{S} \wedge \mathbf{q}^{(j)}\})\mathbf{S}_x - (D^{-1}\{\mathbf{S} \bullet \mathbf{q}_x^{(j)}\})(4A\mathbf{S}) \wedge \mathbf{S}\Big). \quad (3.23)$$

therfore, establishing (3.1b).

Remarks 3.1

(i) $\gamma_0 = x\mathbf{S}$ is a conserved gradient for the LL equation not however in T^*M. It turns out that

$$\tau = \mathbf{S} \wedge \Psi_{LL}(x\mathbf{S}) = x(\mathbf{S} \wedge \mathbf{S}_{xx} + \mathbf{S} \wedge J\mathbf{S}) + \mathbf{S} \wedge \mathbf{S}_x \quad (3.24)$$

is a mastersymmetry of the LL equation.

(ii) In the isotropic limit ($A \rightarrow diag(0,0,0)$), $\Phi_{LL} \rightarrow \Phi_{HM}^2$.

(iii) There exist several equivalent forms of the recursion operator Φ_{LL} and of the second Hamiltonian operator Ω_{LL}. One may verify the Jacobi identity of these equivalent forms by using the approach of Remark 2.1 (iv).

Appendix

In this appendix, we prove that the operator Ω_{HM} given by the formula

$$\Omega_{HM}\mathbf{a} = \mathbf{S} \wedge (\Psi_{HM}\mathbf{a}) = \frac{1}{2}(\mathbf{a}_x - D\{\mathbf{S}D^{-1}(\mathbf{S} \bullet \mathbf{a}_x)\}) \quad (A.1)$$

is a Hamiltonian operator compatible with $\Theta = \mathbf{S}\wedge$.

In the following "$\equiv$" will denote equality up to perfect derivatives.

(i) Ω_{HM} is skew-symmetric:

Consider $\mathbf{a}$, $\mathbf{b}$ in T^*M, then

$$
\begin{aligned}
2(\Omega_{HM}\mathbf{a})\bullet\mathbf{b} &= \mathbf{a}_x\bullet\mathbf{b} - \mathbf{b}\bullet D\{SD^{-1}(\mathbf{S}\bullet\mathbf{a}_x)\}\\
&\equiv -\mathbf{a}\bullet\mathbf{b}_x + (\mathbf{b}_x\bullet\mathbf{S})D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)\\
&\equiv -\mathbf{a}\bullet\mathbf{b}_x - (\mathbf{S}\bullet\mathbf{a}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)\\
&= -\mathbf{a}\bullet\mathbf{b}_x + (\mathbf{a}\bullet\mathbf{S}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)\\
&= -2\Omega_{HM}\mathbf{b}\bullet\mathbf{a},
\end{aligned}
$$

therefore,

$$(\Omega_{HM}\mathbf{a},\ \mathbf{b}) = -(\mathbf{a},\ \Omega_{HM}\mathbf{b}). \tag{A.2}$$

(ii) Ω_{HM} satisfies the Jacobi identity:

Consider **a**, **b**, **c** in T^*M, then

$$
\begin{aligned}
4(\Omega'_{HM}[\Omega_{HM}\mathbf{b}]\mathbf{a})\bullet\mathbf{c} \equiv{}& \{\mathbf{b}_x\bullet\mathbf{c}_x - (\mathbf{S}\bullet\mathbf{c}_x)(\mathbf{S}\bullet\mathbf{b}_x) - (\mathbf{S}_x\bullet\mathbf{c}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)\}D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)\\
&-\{\mathbf{b}_x\bullet\mathbf{a}_x - (\mathbf{S}\bullet\mathbf{a}_x)(\mathbf{S}\bullet\mathbf{b}_x) - (\mathbf{S}_x\bullet\mathbf{a}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)\}D^{-1}(\mathbf{S}\bullet\mathbf{c}_x).
\end{aligned} \tag{A.3}
$$

Therefore, $4(\Omega'_{HM}[\Omega_{HM}\mathbf{b}]\mathbf{a})\bullet\mathbf{c}+$ (cyclic permutations of **a**, **b**, **c**) $\equiv$

$$
\begin{aligned}
&\equiv \{\mathbf{b}_x\bullet\mathbf{c}_x - (\mathbf{S}\bullet\mathbf{c}_x)(\mathbf{S}\bullet\mathbf{b}_x) - (\mathbf{S}_x\bullet\mathbf{c}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)\}D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)\\
&+\{-\mathbf{b}_x\bullet\mathbf{a}_x + (\mathbf{S}\bullet\mathbf{a}_x)(\mathbf{S}\bullet\mathbf{b}_x) + (\mathbf{S}_x\bullet\mathbf{a}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)\}D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)\\
&+\{\mathbf{c}_x\bullet\mathbf{a}_x - (\mathbf{S}\bullet\mathbf{a}_x)(\mathbf{S}\bullet\mathbf{c}_x) - (\mathbf{S}_x\bullet\mathbf{a}_x)D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)\}D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)\\
&+\{-\mathbf{c}_x\bullet\mathbf{b}_x + (\mathbf{S}\bullet\mathbf{b}_x)(\mathbf{S}\bullet\mathbf{c}_x) + (\mathbf{S}_x\bullet\mathbf{b}_x)D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)\}D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)\\
&+\{\mathbf{a}_x\bullet\mathbf{b}_x - (\mathbf{S}\bullet\mathbf{b}_x)(\mathbf{S}\bullet\mathbf{a}_x) - (\mathbf{S}_x\bullet\mathbf{b}_x)D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)\}D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)\\
&+\{-\mathbf{a}_x\bullet\mathbf{c}_x + (\mathbf{S}\bullet\mathbf{c}_x)(\mathbf{S}\bullet\mathbf{a}_x) + (\mathbf{S}_x\bullet\mathbf{c}_x)D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)\}D^{-1}(\mathbf{S}\bullet\mathbf{b}_x) \equiv 0
\end{aligned} \tag{A.4}
$$

(iii) The Hamiltonian operators Ω_{HM} and Θ are compatible i.e. their sum is a Hamiltonian operator.

Since Ω_{HM} and Θ are Hamiltonian operators, it is sufficient to prove that

$$(\{\Omega'_{HM}[\Theta\mathbf{b}]\mathbf{a} + \Theta'[\Omega_{HM}\mathbf{b}]\mathbf{a}\},\ \mathbf{c}) + \text{ cyclic permutations } = 0, \tag{A.5}$$

for any **a**, **b**, **c** in T^*M.
Indeed

$$
\begin{aligned}
-2(\Omega'_{HM}[\Theta\mathbf{b}]\mathbf{a}+\Theta'[\Omega_{HM}\mathbf{b}]\mathbf{a})\bullet\mathbf{c} ={}& (\mathbf{S}\wedge\mathbf{b}\bullet\mathbf{c})(\mathbf{S}\bullet\mathbf{a}_x)+[(\mathbf{S}\wedge\mathbf{b})_x\bullet\mathbf{c}](\mathbf{S}\bullet\mathbf{a}_x)+(\mathbf{c}\bullet\mathbf{S}_x)D^{-1}(\mathbf{S}\wedge\mathbf{b}\bullet\mathbf{a}_x)\\
&-\mathbf{b}_x\wedge\mathbf{a}\bullet\mathbf{c} + (\mathbf{S}\wedge\mathbf{a}\bullet\mathbf{c})\mathbf{S}\bullet\mathbf{b}_x
\end{aligned}
$$

$$\equiv -[(\mathbf{S}\wedge\mathbf{b})_x\bullet\mathbf{c}]D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)-(\mathbf{S}\wedge\mathbf{b}\bullet\mathbf{c}_x)D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)+[(\mathbf{S}\wedge\mathbf{b})_x\bullet\mathbf{c}]D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)$$

$$+D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)(\mathbf{S}\wedge\mathbf{b}\bullet\mathbf{a}_x)-(\mathbf{b}_x\wedge\mathbf{a}\bullet\mathbf{c})-[(\mathbf{S}\wedge\mathbf{a})_x\bullet\mathbf{c}]D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)-(\mathbf{S}\wedge\mathbf{a}\bullet\mathbf{c}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)$$

$$\equiv -(\mathbf{S}\wedge\mathbf{b}\bullet\mathbf{c}_x)D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)+(\mathbf{S}\wedge\mathbf{b}\bullet\mathbf{a}_x)D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)-(\mathbf{b}_x\wedge\mathbf{a}\bullet\mathbf{c})$$

$$-(\mathbf{S}\wedge\mathbf{a}_x\bullet\mathbf{c})D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)-\mathbf{S}\wedge\mathbf{a}\bullet\mathbf{c}_xD^{-1}(\mathbf{S}\bullet\mathbf{b}_x). \tag{A.6}$$

So

$$2(\Omega'_{HM}[\Theta\mathbf{b}]\mathbf{a}+\Theta'[\Omega\mathbf{b}]\mathbf{a})\bullet\mathbf{c}+\text{ cyclic permutations of }\mathbf{a},\mathbf{b},\mathbf{c}\equiv$$

$$\equiv \mathbf{b}_x\wedge\mathbf{a}\bullet\mathbf{c}+(\mathbf{S}\wedge\mathbf{b}\bullet\mathbf{c}_x)D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)-(\mathbf{S}\wedge\mathbf{b}\bullet\mathbf{a}_x)D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)+(\mathbf{S}\wedge\mathbf{a}_x\bullet\mathbf{c})D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)+(\mathbf{S}\wedge\mathbf{a}\bullet\mathbf{c}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)$$

$$+\mathbf{c}_x\wedge\mathbf{b}\bullet\mathbf{a}+(\mathbf{S}\wedge\mathbf{c}\bullet\mathbf{a}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)-(\mathbf{S}\wedge\mathbf{c}\bullet\mathbf{b}_x)D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)+(\mathbf{S}\wedge\mathbf{b}_x\bullet\mathbf{a})D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)+(\mathbf{S}\wedge\mathbf{b}\bullet\mathbf{a}_x)D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)$$

$$+\mathbf{a}_x\wedge\mathbf{c}\bullet\mathbf{b}+(\mathbf{S}\wedge\mathbf{a}\bullet\mathbf{b}_x)D^{-1}(\mathbf{S}\bullet\mathbf{c}_x)-(\mathbf{S}\wedge\mathbf{a}\bullet\mathbf{c}_x)D^{-1}(\mathbf{S}\bullet\mathbf{b}_x)+(\mathbf{S}\wedge\mathbf{c}_x\bullet\mathbf{b})D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)+(\mathbf{S}\wedge\mathbf{c}\bullet\mathbf{b}_x)D^{-1}(\mathbf{S}\bullet\mathbf{a}_x)$$

$$=(\mathbf{b}\wedge\mathbf{a}\bullet\mathbf{c})_x\equiv 0$$

Acknowledgements

One of the authors (A.S.F.) is grateful to J. Keller and his group at the Department of Mathematics, for their hospitality during the author's sabbatical leave at Stanford University. This work was supported in part by the Air Force Office of Scientific Research under grant #87-0310 and the National Science Foundation under grant #DMS-8501325.

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Resist Development Described by Least Action Principle-Line Profile Prediction*

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Abstract

A new idea is introduced requiring that each development path will be the path of least resistance to developer penetration. Consequently, minimum dissolution time is required for the development of the final line profile. This idea manifests itself in a variational calculation of the path integral along each local development trajectory, from which the dissolution profile is obtained uniquely, as a solution of a non-linear PDE. The PAC concentration is obtained from the standard Dill's equations for the exposure-bleaching process for both monotonic as well as standing waves. The procedure has been implemented and tested. It has been found to be very accurate and it eliminates the path crossings inherent in the predictions of the string algorithm. The arbitrary elimination of unfavorable points is avoided as well for all developing times.

* Supported in part by Grants AFOSR-87-0310 and NSF #ECS-8611298

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I. Introduction

The importance of simulations for VLSI lithography and etching processes is widely accepted and many simulation techniques are utilized throughout the IC industry. The most cor mon simulation systems are SAMPLE (1) and PROLITH (2) which are the established standards. Both SAMPLE and PROLITH allow the user to search for optimal conditions for an experiment at hand. Both combine a projection exposure model for a thin photoresist film with a "development" model, and an ultimate goal of both systems is an accurate prediction of the line profile over any substrate topography.

The "exposure" model is a system of coupled, non-linear partial differential equations first proposed by Dill (3). The two unknown functions are $M(x,z,t)$, the photoactive compound concentration (PAC), and $I(x,z,t)$, the intensity of light at coordinates (x,z) at time t. One should note that the order of the equation determining $I(x,z,t)$ or the corresponding electric field $E(x,z,t)$ is either first or second depending on whether the film is thick (no standing waves present) or thin (standing waves are a dominant feature). The second equation is a first order rate equation, expressing the assumption that the rate of change of log M is declining and it is proportional to the light intensity I, with initial condition $M(x,z,0) = 1$.

The monotonic case has been solved analytically (4) and the solution has been used in various applications (5, 6). The standing waves case has been solved exactly (7) but the solution is very complicated and a WKB approximation scheme has been proposed (8) to replace the standard iteration schemes.

In this paper we assume that the PAC concentration $M(x,z)$ is a given function that has been obtained by one of the above methods, after an exposure time t_f. Here we concentrate on the etching-development model. Various authors (9–13) offered phenomenological dissolution rate-development functions $R(M)$, that in essence represent the velocity of dissolution of the exposed PAC.

Both simulators (SAMPLE, PROLITH) employ the "string development algorithm" in their development-etching model . The boundary between the developed and undeveloped regions is expressed as series of points in the xz-plane connected by linear splines (a string). Each point advances along the angle bisector of the two adjacent segments, with a velocity $R(x, z)$. As the density of points increases in some locations several are eliminated, and others are introduced in sparse regions, so their density along the string remains roughly constant. When the final development time is achieved the programs report the final simulated profile.

It is the purpose of this paper to propose an alternative to this string algorithm, based on least square action principle. It has several advantages over the string-algorithm:

(i) It is mathematically rigorous.

(ii) The proposed method is applicable to three dimensions - a serious limitation of the string algorithm.

(iii) It contains no arbitrary additions and subtractions of points along the profile.

(iv) There are no crossings of development paths that create loops in the profile. These loops are present in the standard simulations, creating the necessity to delete them.

II. Propagation of a disturbance

Let a disturbance propagate through a medium with velocity $R[M(x, z)]$. The disturbance propagates orthogonal to itself. In other words, at time t, one must obtain

$$| \text{ grad } t | = R^{-1} \tag{1}$$

or more precisely (for the standard case)

$$\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2 = \frac{1}{R^2[M(x, z)]} \tag{2}$$

This is a non-linear first order PDE which can be solved by the method of envelope-characteristics.

It can be shown that as long as a "ray" *does not cross* any other development "ray", its $x(s)$ and $z(s)$ coordinates as functions of the arclength along the ray are determined by the following system of ordinary differential equations:

$$\frac{d^2x}{ds^2} = \frac{\partial logR}{\partial x}\left(\frac{dx}{ds}\right)^2 + \frac{\partial logR}{\partial z}\left(\frac{dx}{ds}\right)\left(\frac{dz}{ds}\right) - \frac{\partial logR}{\partial x} \qquad (3a)$$

$$\frac{d^2z}{ds^2} = \frac{\partial logR}{\partial x}\left(\frac{dx}{ds}\right)\left(\frac{dz}{ds}\right) + \frac{\partial logR}{\partial z}\left(\frac{dz}{ds}\right)^2 - \frac{\partial logR}{\partial z} \qquad (3b)$$

The standard formulation of development time t is given by

$$t = \int_o^1 \frac{ds}{R[M\{x(s), z(s)\}]} \qquad (4)$$

The variation of t, i.e. δt, resulting from a slight development $\delta x, \delta z$ leads precisely to equations (3a) and (3b). In other words t as given by equation (4) solves the non-linear PDE (2) inside its envelope.

This formulation (introduced by Carrier & Pearson) dictates the algorithm we use.

(i) Obtain an initial profile

(ii) Develop each point for a time interval Δt using the system of equations (3a, 3b), and make sure that the paths do not cross by selecting Δt to be small enough

(iii) Use the new profile as the initial profile and repeat the process.

The time interval Δt is dependent on the curvature of the profile, since it determines the thickness of the characteristic strip. Note that we are dealing with a *local* process and that the individual rays may not cross, since the physical process is unique and smooth and crossing rays would lead to either shock-waves or a non unique solution. Thus the strips must be dealt with on an infinitesimal level and not globally.

III. Implementation and Examples

The initial profile is taken as the x-axis, with 51 equally spaced points. A specialized Runge-Kutta scheme was developed for a system of five ordinary differential equations that include the two coordinates, their arclength derivatives and the development time. For most processes tested, the average development time step, Δt, was found to be 0.05 sec. After each time step an optimizing cubic spline routine was implemented, resulting in a smooth representation of the profile. The new profile is divided into segments of equal arclength in order to maintain consistency with the previous profile. Thus the number of segments varies according to the shape of the profile. This process is repeated until the prescribed development time has elapsed. We refer to this program by the name EIKPCS.

It should be emphasized that the description of a three dimensional profile must be determined parametrically. However, the cases reported here can be expressed explicitly as functions of the coordinates. In these cases the profile is reported in various segments, where in each segment the corresponding functions are single-valued. These segments are connected to represent the final etching profile.

The dissolution rate function $R(M)$ employed in this study is the one proposed by C. Mack(9). Throughout this paper the following development parameters were used: $R_{max} = 200$ nm/s, $R_{min} = 1$ nm/s, $m_{TH} = 0.5$, $n = 5$. Figure 1 illustrates the relative development rate as a function of the relative PAC concentration.

As described in the introduction two data files of $M(x,z)$ values are utilized, RM1 and EXPOSE. The file RM1 has been generated to simulate the CEM-positive resist system proposed by Mack(14), which corresponds to the monotonic example illustrated in this paper. The exact solutions of the Dill's model equations(4) were used in this simulation. The file EXPOSE was given to us by C. Mack, and it corresponds to a standard standing waves example in PROLITH. Both of these data files are used for demonstration purposes only.

In fig. 2,3 we compare the dissolution profile obtained from the string algorithm for the monotonic case to the results obtained from the proposed algorithm, employing the file RM1. In these figures 60 sec of development time at 0.05 sec per time step was simulated. The program of the string algorithm has the "loop eliminator" routine turned off. The reader should observe the early formation of a loop at the upper corner, while the EIKPCS profile does not exhibit this aberration. In fig. 4,5 we make a similar comparison at 75 sec development time, and the loop is clearly demonstrated.

In fig. 6,7,8 we display the utility of our system to handle standing waves using the data file EXPOSE. In these examples, development times of 30 sec, 45 sec and 60 sec were employed. The final profiles *do not* exhibit any loop. They contain approximately 230 points and as the resolution increases they can be made smoother. It is well-known that the standing waves systems display several sizeable loops when the string algorithm is employed and when these loops are eliminated they tend to give the impression of somewhat reduced amplitude.

IV. Conclusion

We conclude that the mathematically rigorous algorithm indeed performs as expected, thus reducing the ambiguity in development simulation.

Acknowledgement

The authors extend their thanks to Chris Mack for his interest in this work, his stimulating discussions and the supply of his data file EXPOSE. A stimulating discussion with A. Neureuther is gratefully acknowledged.

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Figure Captions

Figure 1: Relative development rate vs. relative PAC concentration using MACK's model. The development parameters are $R_{max} = 200nm/s$, $R_{min} = 1nm/s$, $m_{TH} = 0.5$ and $n = 5$.

Figure 2: Simulated resist profile of 60 sec development time using the program EIKPCS developed in this work and the data file RM1

Figure 3: Simulated resist profile of 60 sec development time using the string algorithm and the data file RM1

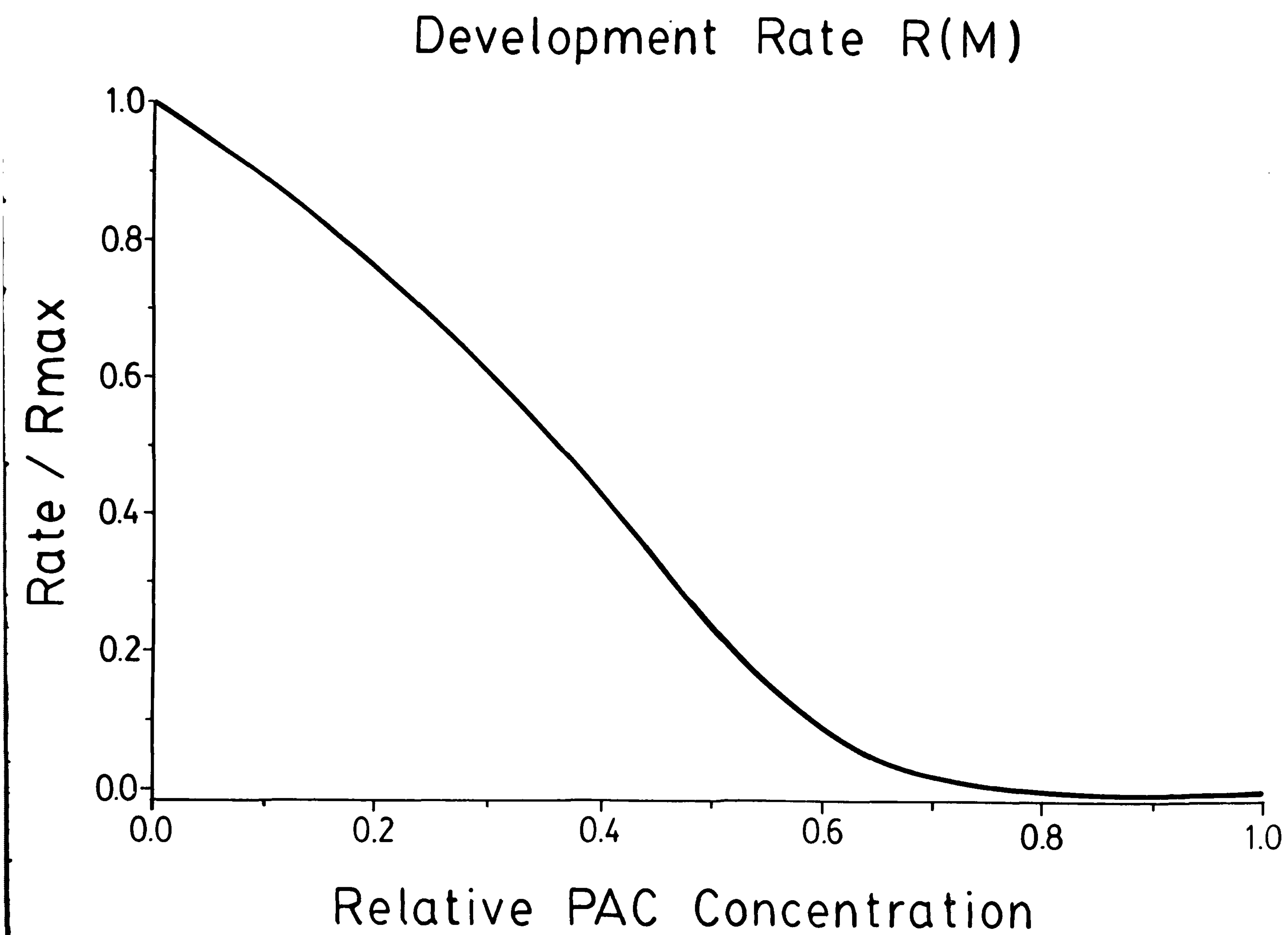
Figure 4: Simulated resist profile of 75 sec development time using the program EIKPCS developed in this work and the data file RM1

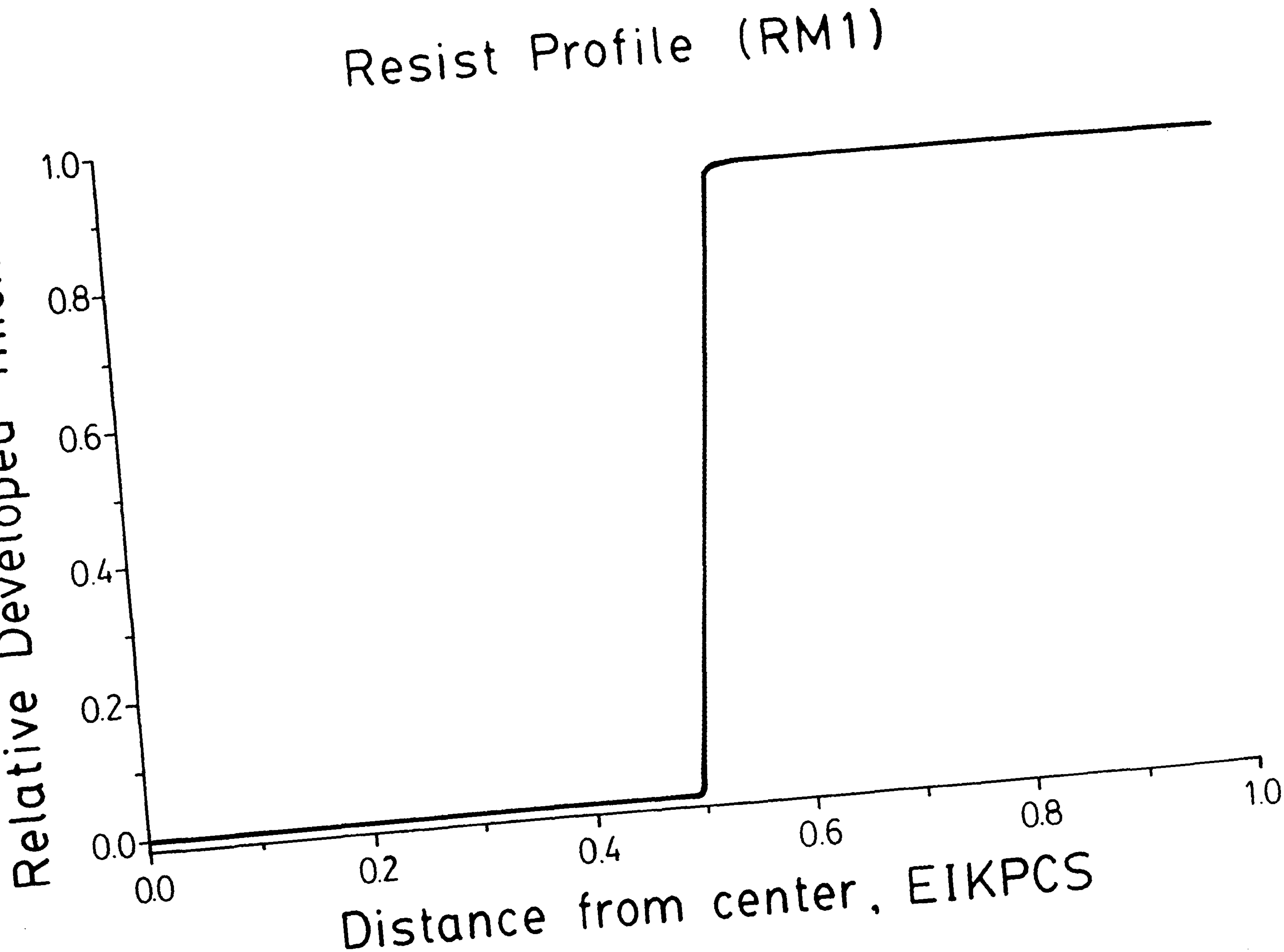
Figure 5: Simulated resist profile of 75 sec development time using the string algorithm and the data file RM1

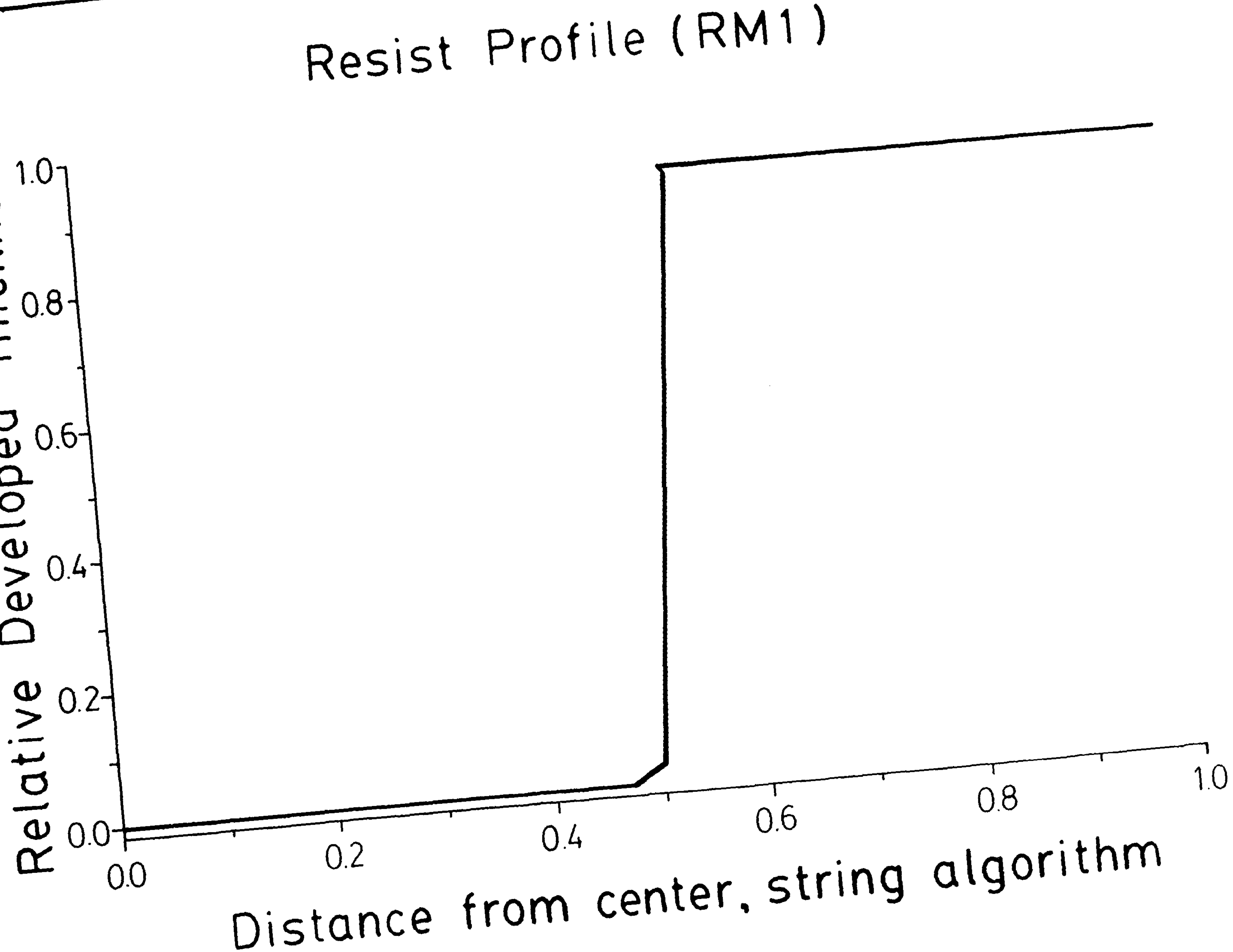
Figure 6: Simulated dissolution profile of a photoresist with reflecting substrate, using the data file EXPOSE of C.Mack and the program EIKPCS at 30 sec development time

Figure 7: Simulated dissolution profile of a photoresist with reflecting substrate, using the data file EXPOSE of C.Mack and the program EIKPCS at 45 sec development time

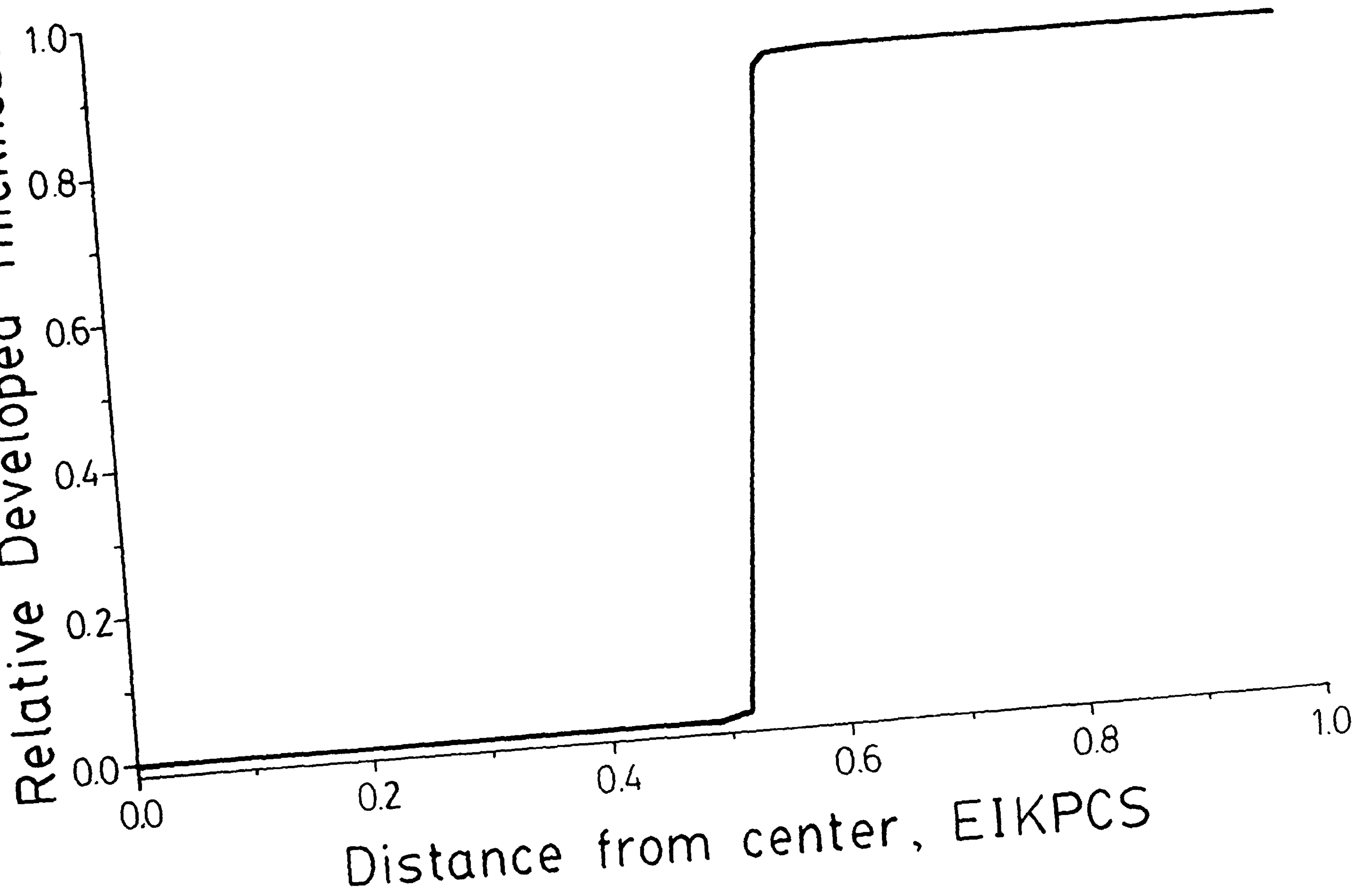
Figure 8: Simulated dissolution profile of a photoresist with reflecting substrate, using the data file EXPOSE of C.Mack and the program EIKPCS at 60 sec development time

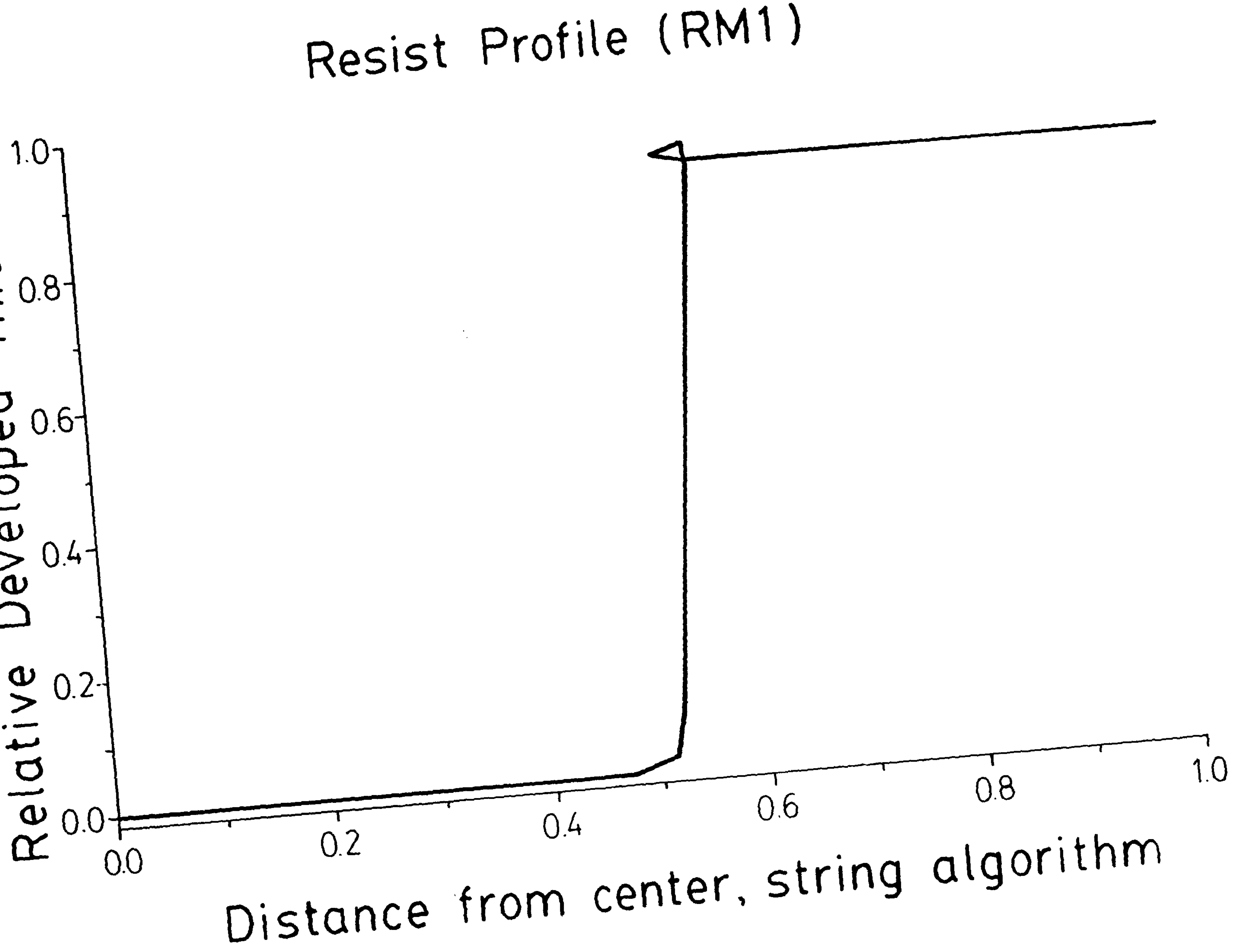

Development Rate R(M)
Rate / Rmax
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Relative PAC Concentration

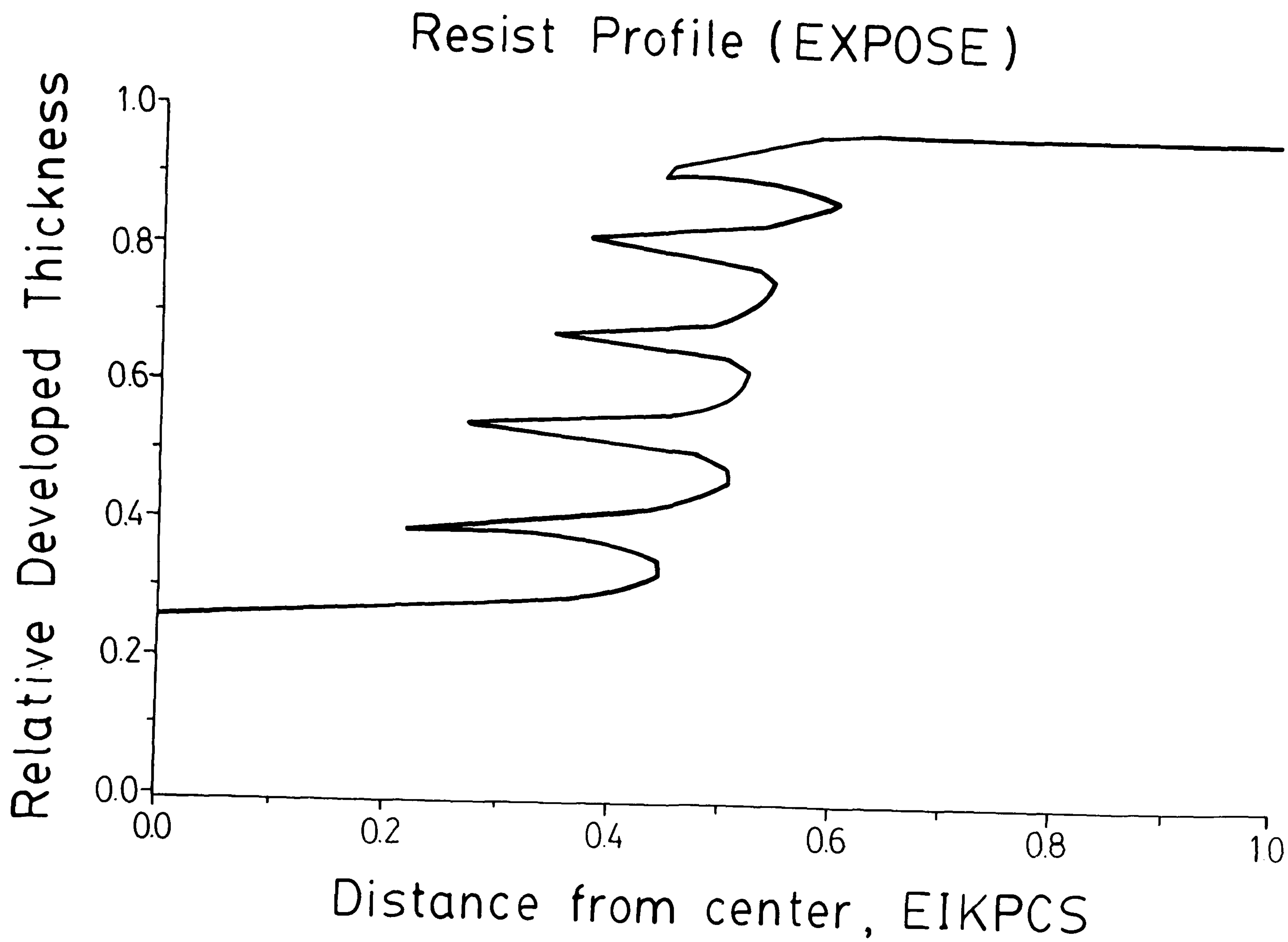

Resist Profile (RM1)
Relative Developed Thickness
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Distance from center, EIKPCS

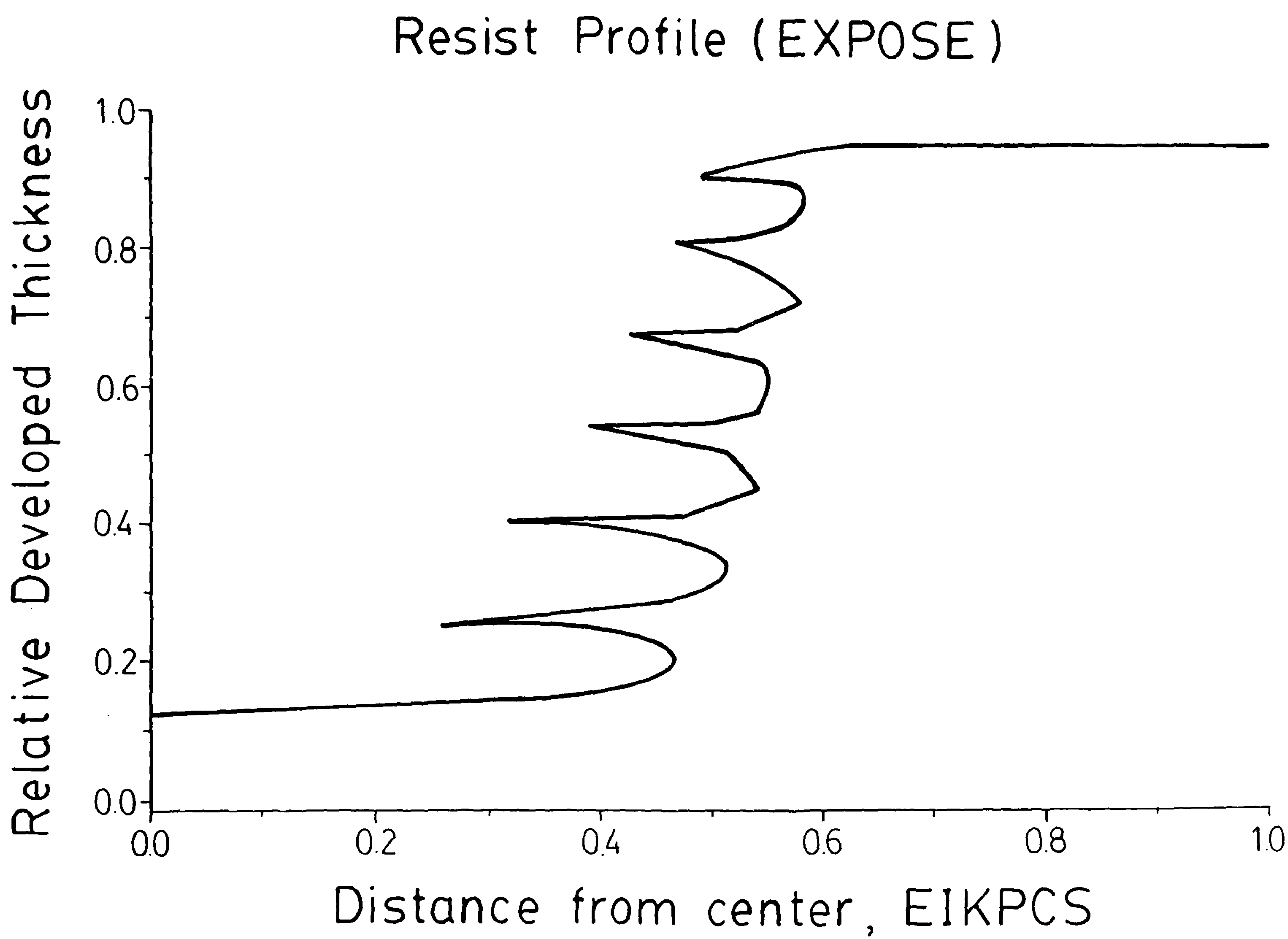

Resist Profile (RM1)
Relative Developed Thickness
0.0
0.2
0.4
0.6
0.8
1.0
Distance from center, string algorithm

Resist Profile (RM1)

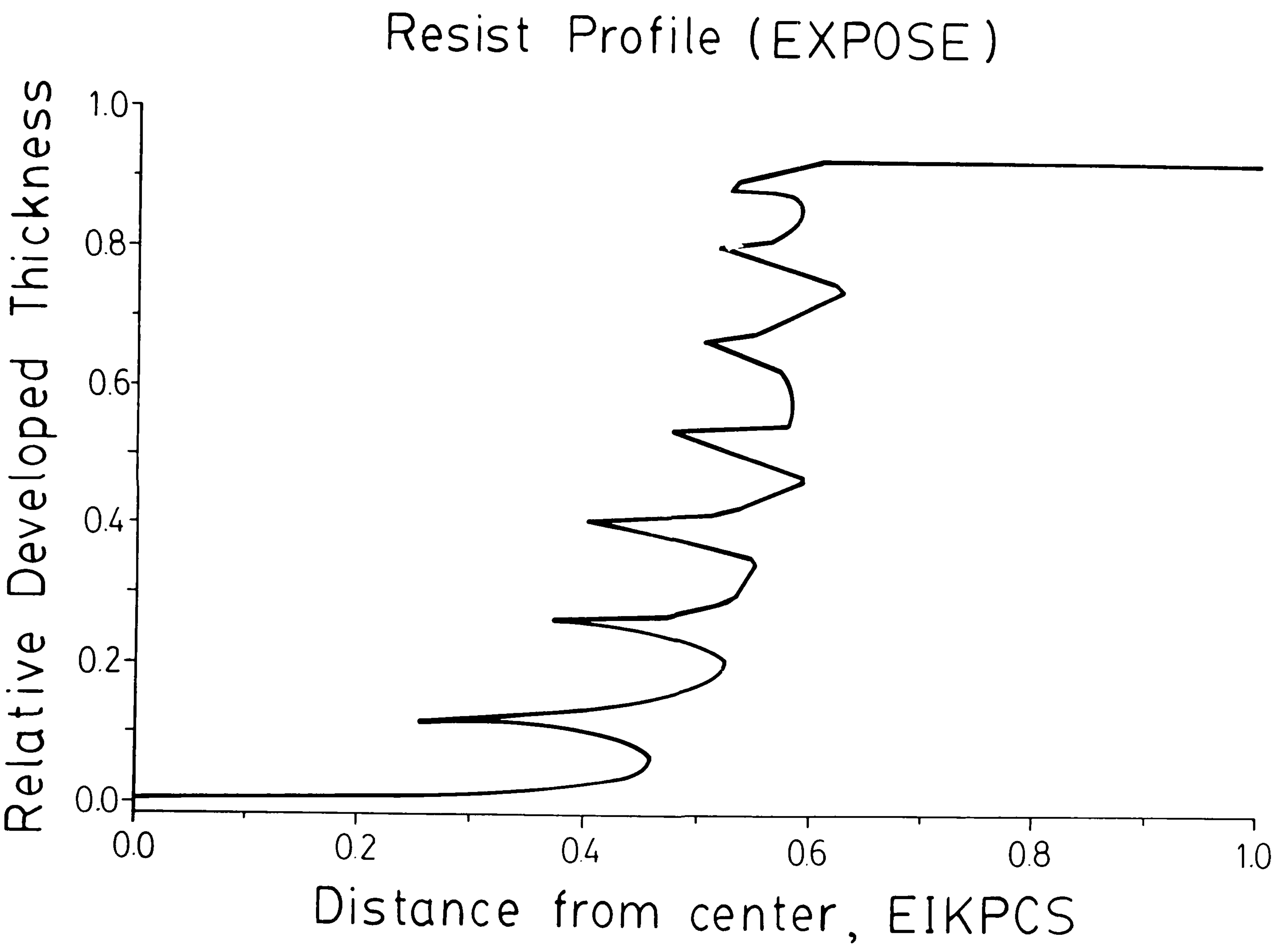



Resist Profile (RM1)
Relative Developed Thickness
0.0
0.2
0.4
0.6
0.8
1.0
Distance from center, string algorithm


Resist Profile (EXPOSE)
Relative Developed Thickness
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Distance from center, EIKPCS



Resist Profile (EXPOSE)
Relative Developed Thickness
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Distance from center, EIKPCS



Resist Profile (EXPOSE)
Relative Developed Thickness
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Distance from center, EIKPCS

An Initial-Boundary Value Problem for the Nonlinear Schrödinger Equation

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December 1987

Abstract

We present a method for studying initial-boundary value problems associated with integrable nonlinear evolution equations. For concreteness we consider the nonlinear Schrödinger equation in the variable $q(x,t)$, x in $[0,\infty)$, with a mixed boundary condition, i.e. $q_x(0,t) + \alpha q(0,t)$ is given (α is an arbitrary real constant). $q(x,t)$ can be obtained by solving a linear integral equation uniquely defined in terms of appropriate scattering data. These data satisfy a single nonlinear integrodifferential equation uniquely defined in terms of the boundary condition. For the special case of a homogeneous boundary condition, the scattering data is found in closed form.

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1 Introduction

It is well known that the inverse scattering transform (IST) has been applied to a large number of physically important nonlinear evolution equations in 1+1 (i.e. in one spatial and one temporal dimensions). The initial value problems for decaying [1], periodic [2], and self similar potentials [3] has received much attention. The IST has also been successfully extended to initial value problems for decaying potentials for equations in 2+1 (i.e. in two spatial and one temporal dimensions) [4].

In spite of the above success, the question of extending the IST to solve initial-boundary value problems remains essentially open [5]. The simplest such problem arises if an equation is formulated on the half-infinite line. Let us consider the nonlinear Schrödinger equation (NLS)

$$iq_t = q_{xx} \pm 2|q|^2 q, \quad 0 \le x < \infty; \quad q(x,0) = h(x), \quad q_x(0,t) + \alpha q(0,t) = g(t), \tag{1.1}$$

where $h(x)$ decays for large x, α is a real constant, and the given functions $h(x)$, $g(t)$ have appropriate smoothness, and satisfy the necessary compatibility conditions to ensure the existence of solution at $x = 0$, $t = 0$. Solving such an initial-boundary value problem has important physical and mathematical implications:

(i) Most physical problems are naturally formulated as boundary value problems. For example, injecting current in a neuron, or sending optical solitons down a monomode fiber are boundary value problems. In particular, NLS with an additional term q_x on the right hand side and $\alpha \to \infty$, models water waves [1]. Equation (1.1) also arises in the propagation of optical solitons [6], as well as in several other important physical problems. Since NLS usually arises in applications in non-laboratory coordinates it is useful to consider equation (1.1) with $\alpha \neq 0$.

(ii) The linear limit of the standard IST (where $-\infty < x < \infty$, $q(x,0)$ given) is the Fourier transform, which is why the IST is considered as the nonlinear analogue of the Fourier transform [7]. The linear limit of (1.1), i.e.

$$iq_t = q_{xx}, \quad 0 \le x < \infty; \quad q(x,0) = h(x), \quad q_x(0,t) + \alpha q(0,t) = g(t), \tag{1.2}$$

can be solved by the sine transform ($\alpha \to \infty$), or the cosine transform ($\alpha = 0$), or in general by the transform [8]

$$\hat{q}(k) \doteq \int_0^\infty d\xi (e^{2ik\xi} + f(k)e^{-2ik\xi}) q(\xi) d\xi, \qquad f(k) = \frac{2ik+\alpha}{2ik-\alpha},$$

$$q(x) = \frac{1}{\pi}\int_0^\infty dk (e^{-2ikx} + f(-k)e^{2ikx})\hat{q}(k) + 2\alpha e^{-\alpha x}\int_0^\infty d\xi e^{-\alpha\xi} q(\xi), \tag{1.3}$$

where the second term of (1.3b) is missing if $\alpha < 0$. It is thus natural to ask what is the nonlinear analogue of the above transforms.

In this paper we present a method for studying boundary value problems in $1+1$, and we apply this method to equation (1.1):

(i) The first step involves finding the correct x-transform of the given nonlinear equation. Indeed, our formalism in the linear limit, i.e. q small, reduces to the inversion formula (1.3).

(ii) The next step involves finding the evolution of the scattering data. In the linear case it corresponds to using the transform (1.3) to solve $iq_t = q_{xx}$. In the usual IST one uses the t-part of the Lax pair to find the evolution of the scattering data. However, in our case $\Psi_{tx} - \Psi_{xt} \neq 0$, where $\Psi(x,t,k)$ is the eigenfunction appearing in the Lax pair, and one needs first to obtain the correct t-part. In this respect we use the given evolution equation and an *integral* (as opposed to the usual differential) representation of the x-part of the Lax-pair (hence we do not need apriori knowledge of $\Psi_{tx} - \Psi_{xt}$). An alternative way to finding the t-part of the Lax pair is to use that $\Psi(x,t)$ is continuous at $x=0$. Having obtained the t-part of the Lax pair, the evolution of the scattering data follows. The correct choice of the x-transform is reflected by the fact that the evolution of the scattering data depends on $g(t)$ and not on $q_x(0,t)$, $g(0,t)$ separately. Furthermore, in the linear limit the scattering data satisfies

$$\hat{q}_t - 4ik^2\hat{q} = \frac{-4k}{2ik-\alpha}(q_x(0,t) + \alpha q(0,t)), \tag{1.4}$$

which is precisely the time evolution of the linear transform (1.3) when applied to equation (1.2). However, the above evolution also depends on certain quadratic products of $\Psi(0,t,k)$.

(iii) The final step consists of expressing these quadratic products in terms of the scattering data. This yields a nonlinear equation for the scattering data. In the case of the NLS equation (1.1), this yields the following nonlinear singular integro-differentiation equation for the reflection coefficient $\bar{b}(\alpha \to \infty)$

$$\frac{\bar{b}_t - 4ik^2\bar{b}}{4k} = -q(0,t) + \int_{-\infty}^{\infty} dk' \frac{\frac{\bar{b}(k')}{8k'} - \frac{\bar{b}(k)}{8k}}{k'-k} \frac{\partial}{\partial t} H \ln\left(1 \pm |\bar{b}|^2\right)(k'). \tag{1.5}$$

(a) The application of the above method to other equations in $1+1$ has certain analytical complications reflecting difficulties with the linearized version of the given equation. However, it can be applied to other equations in $1+1$, as well as in $2+1$.

(b) This method opens the way for studying boundary value problems on finite domains.

(c) It can be used to study forced integrable systems where the forcing involves Dirac's delta function and its derivatives.

The special cases of $q(x,0) = 0$ and $q_x(x,0) = 0$ were considered in [9]. Also for the case of a general homogeneous boundary condition, i.e. $q_x(x,0) + \alpha q(x,0) = 0$, Sklyanin has

established complete integrabilities by proving the existence of infinitely many conservation laws [10].

A. Outline and Open Questions

In §II we consider the NLS with $q(0,t)$ given, i.e. we study the nonlinear analogue of the sine transform. If $q(0,t)=0$ the analysis is straightforward: This problem is equivalent to one formulated in $-\infty < x < \infty$ [9] and can be solved in terms of a system of linear integral equations. If $q(0,t) \neq 0$ the analysis becomes nonlinear. The main result of this section is expressed by proposition 2.10: The problem is again formulated in terms of a system of linear integral equations uniquely defined in terms of appropriate scattering data. However, while these scattering data are found in closed form if $q(0,t)=0$, they satisfy a nonlinear singular integrodifferential equation if $q(0,t) \neq 0$ (see (2.45)). The existence and uniqueness of solutions of this nonlinear equation remains open. Throughout this section we assume that the transmission coefficients $\frac{1}{\bar{a}(k)}, \frac{1}{a(k)}$ do not have poles in the upper, lower half k-complex plane (see (c) below).

In §3 we consider the general case where $q_x(0,t)+\alpha q(0,t)$ is given. The two main results of this section are:

(i) If $q_x(0,t)+\alpha q(0,t)=0$ the problem is equivalent to one for $-\infty < x < \infty$ and can be solved via a system of linear integral equations uniquely defined in terms of appropriate scattering data; these data are found in *closed form*.

(ii) If $q_x(0,t)+\alpha(0,t) \neq 0$ the problem is nonlinear since the scattering data again satisfy a nonlinear singular integrodifferential equation. The evolution of the scattering data is given explicitly and involves $q_x(0,t)+\alpha q(0,t)$. For brevity of presentation, the details of how to derive the analogue of (2.45) are omitted. We again assume that $\frac{1}{\bar{a}}$, $\frac{1}{a}$ do not have poles.

Several important problems remain open:

(a) The uniqueness and existence of solutions of the nonlinear singualr integrodifferential equation satisfied by the scattering data needs to be established.

(b) The question of whether the above equation can be linearized remains open. This question is important not only for practical but also for theoretical reasons: It has been assumed so far that complete integrability is a local property. However, if the above equation can not be linearized, it would be implied that integrability also depends on the boundary conditions.

(c) The formalism presented here can be modified to include poles of the transmission coefficients. However, since these poles move in time, the analysis becomes quite more complicated. Preliminary results indicate that it might be possible to avoid considering directly these poles by mapping the given initial and boundary data to suitable data which do not possess poles. We have found [17] that $t \to -t$ and $q \to q^*$ are useful transformations in this respect.

(d) The existence and uniqueness of solution for the Korteweg-deVries (KdV) equation for $0 \leq x < \infty$, where $q(x,0)$ and $q(0,t)$ are given, has been proven by Bona and Winther [11]. The application of our method to the KdV equation has certain difficulties stemming from the fact that even the application of the x-transform to solving the linearized KdV is problematic. This suggests that perhaps one needs to study the nonlinear analogue of the Laplace transform. We expect that some of the ideas presented here will also be useful for this study of this problem as well.

(e) It has been established numerically that the KdV with $0 \leq x < \infty$, $q(x,0)$ given, can be used to generate solitons (see the discussion by Keller [12] for details). Similar results have recently been found for the NLS [16]. An asymptotic investigation of the nonlinear equation mentioned above (equation (1.5)), taking into consideration (c), should provide some insight into these numerical observations as well as should yield the appropriate mathematical formulae.

2 Dirichlet Boundary Condition

We first consider (1.1) with $\alpha \to \infty$. The linear analogue of this problem is given by

$$iq_t = q_{xx}, \quad 0 \leq x < \infty, \quad q(x,0) = h(x), \quad q(0,t) \text{ given}, \tag{2.1}$$

and can be solved by the sine transform,

$$\hat{q}(k,t) \doteq \int_0^\infty d\xi q(\xi,t) sink\xi, \quad q(x,t) = \frac{2}{\pi}\int_0^\infty dk\hat{q}(k,t) sinkx, \tag{2.2}$$

where the sine data satisfies

$$\hat{q}_t = ik^2\hat{q} - ikq(0,t). \tag{2.3}$$

Alternatively, one may solve (2.1) by the Fourier transform, by embedding (2.1) in $-\infty < x < \infty$; this can be achieved by using an odd extension, then (2.1) is equivalent to

$$i\tilde{q}_t = \tilde{q}_{xx} - 2q(0,t)\delta'(x), \quad -\infty < x < \infty, \quad q(x,0) = h(x), \tag{2.4}$$

$$\tilde{q}(x,t) \doteq q(x,t)H(x) - q(-x,t)H(-x),$$

where $H(x)$ denotes the Heaviside function, i.e. $H(x) = 1$, if $x > 0$, $H(x) = 0$ if $x < 0$, and $\delta'(x)$ denotes the derivative of the Dirac distribution.

Similar considerations apply to the nonlinear problem at hand, which also can be embedded in $-\infty < x < \infty$ by employing distributions (the details are given in [13]). Here we use an odd extension of $q(x,t)$ in order to derive the nonlinear analogue of the sine transform, but we avoid the explicit use of distributions.

2.1 The Nonlinear Analogue of the Sine Transform

The first step of our method involves finding the correct x-transform for the nonlinear equation (1.1). This amounts to using the x-part of the Lax pair to derive an inversion formula which reduces to (2.2) for small q.

A. Analytic Eigenfunctions

Let us consider the linear eigenvalue problem

$$\varphi_x = ikJ\varphi + \tilde{Q}\varphi, \quad -\infty < x < \infty, \quad \tilde{Q} \doteqdot Q(x)H(x) - Q(-x)H(-x), \tag{2.5}$$

$$J \doteqdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 0 & \tilde{q}(x) \\ \tilde{r}(x) & 0 \end{pmatrix},$$

where φ is a 2×2 matrix valued function of x. Let $\varphi = \Phi exp(ikxJ)$ then (2.5) becomes

$$\Phi_x = ik[J, \Phi] + \tilde{Q}\Phi, \tag{2.6}$$

where [,] denotes the usual commutator.

Proposition 2.1. Let the matrices Ψ, Φ solve

$$\Psi = I - \int_x^{\infty} d\xi e^{ik(x-\xi)\hat{J}} \tilde{Q}\Psi, \quad \Phi = I + \int_{-\infty}^{x} d\xi e^{ik(x-\xi)\hat{J}} \tilde{Q}\Phi, \tag{2.7}$$

where if F is an arbitrary 2×2 matrix and if Y is a diagonal matrix, then $exp(\hat{Y})F \doteqdot exp(Y)Fexp(-Y)$. Then

(i) Ψ, Φ solve (2.6).

(ii) $\Psi = (\Psi^-, \Psi^+)$, $\Phi = (\Phi^+, \Phi^-)$, where $+(-)$ denotes analyticity in the upper(lower) half k-complex plane.

B. The Scattering Equation

Proposition 2.2.

(i) The eigenfunctions Ψ, Φ defined by (2.7) are related via

$$\Psi(x,k) = \Phi(x,k)e^{ikx\hat{J}}S(k), \quad S(k) \doteqdot \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \doteqdot I - \int_{-\infty}^{\infty} d\xi e^{-ik\xi\hat{J}} \tilde{Q}(\xi)\Psi(\xi,k). \tag{2.8}$$

(ii) $\bar{a}(k)$, $a(k)$ are $+,-$ functions respectively. (2.9)

(iii) $detS(k) = 1$ (2.10)

(iv) $\Psi(-x,-k) = \Phi(x,k)$ (2.11)

(v) $S(k)S(-k) = I$, or $\bar{a}(-k) = a(k)$, $\bar{b}(-k) = -\bar{b}(k)$, $b(-k) = -b(k)$. (2.12)

Proof

(i) Since both Ψ, Φ satisfy (2.6) they are related via $\Psi = \Phi exp(ikx\hat{J})C$, where C is an x-independent matrix. The form of C follows by considering the above equation as $x \to -\infty$.

(ii) Follows from the definitions of S, Ψ.

(iii) Follows from the facts that $det\Psi = det\Phi = 1$.

(iv) Follows from the fact that $\tilde{Q}(-x) = -\tilde{Q}(x)$.

(v) Follows from (2.8) and (2.11).

C. The Large k Behavior

The potential $\tilde{Q}(x)$ is discontinuous at the origin. Thus we expect that the scattering data decay slowly for large k.

Proposition 2.3. Let k be real. Then

(i) $a \to 1, \;\; \bar{a} \to 1, \;\; \bar{b} \to \frac{q(0)}{ik}, \;\; b \to -\frac{r(0)}{ik}$ as $k \to \infty$. (2.13)

(ii) Let $x > 0$, then

$$\Psi \longrightarrow \begin{pmatrix} 1 & \frac{q(x)}{2ik} \\ -\frac{r(x)}{2ik} & 1 \end{pmatrix}, \quad \Phi \longrightarrow \begin{pmatrix} 1 & \frac{q(x)}{2ik} - \frac{q(0)e^{-2ikx}}{ik} \\ -\frac{r(x)}{2ik} + \frac{r(0)e^{2ikx}}{ik} & 1 \end{pmatrix} \quad \text{as } k \to \infty. \tag{2.14}$$

Proof

(i) For large $k, \Psi \to I$, thus $a \to 1, \;\; \bar{a} \to 1, \;\; \bar{b} \to -\int_{-\infty}^{\infty} d\xi \tilde{q} e^{2ikx} = \int_0^\infty d\xi q(\xi)(e^{-2ik\xi} - e^{2ik\xi})$. Integrating the last equation by parts we obtain $\bar{b} \to \frac{q(0)}{ik}$.

(ii) Equation (2.14a) follows from (2.7a). To obtain (2.14b) note that (2.7b) implies

$$\begin{pmatrix} \Phi_1^+ & \Phi_1^- \\ \Phi_2^+ & \Phi_2^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_{-\infty}^{x} d\xi \begin{pmatrix} \tilde{q}\Phi_2^+ & \tilde{q}\Phi_2^- e^{-2ik(x-\xi)} \\ \tilde{r}\Phi_1^+ e^{2ik(x-\xi)} & \tilde{r}\Phi_1^- \end{pmatrix}.$$

Thus $\Phi_1^+ \to 1, \;\; \Phi_2^- \to 1$ and

$$\Phi_1^- \to -\int_{-\infty}^{0} d\xi q(-\xi) e^{-2ik(x-\xi)} + \int_0^x d\xi q(\xi) e^{-2ik(x-\xi)} \longrightarrow \frac{e^{-2ikx}}{2ik}\left[e^{2ikx} q(x) - 2q(0)\right].$$

D. The Inversion Formula

Proposition 2.4. Let $a, b, \bar{a}, \bar{b}$ be defined by (2.8).

(i) Assume that the vectors Φ^+, Φ^- solve

$$\Phi^+ = \binom{1}{0} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{\frac{b}{a}(k') e^{2ik'x}\Phi^-(k')}{k' - (k + i0)},$$

$$\Phi^- = \binom{0}{1} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{\frac{\bar{b}}{\bar{a}}(k')e^{-2ik'x}\Phi^+(k')}{k' - (k - i0)}, \quad x > 0. \tag{2.15}$$

Then

$$q(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{b}{a} e^{-2ikx}\Phi_1^+, \quad r(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{b}{a} e^{2ikx}\Phi_2^-, \quad x > 0. \tag{2.16}$$

(ii) Assume that the vector Ψ^+, Ψ^- solve

$$\Psi^- = \binom{1}{0} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{\frac{b}{\bar{a}}(k')e^{2ik'x}\Psi^+(k')}{k' - (k - i0)}$$

$$\Psi^+ = \binom{0}{1} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{\frac{\bar{b}}{a}(k')e^{-2ik'x}\Psi^-(k')}{k' - (k + i0)}, \quad x > 0. \tag{2.17}$$

Then

$$q(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{\bar{b}}{a} e^{-2ikx}\Psi_1^-, \quad r(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{b}{\bar{a}} e^{2ikx}\Psi_2^+, \quad x > 0. \tag{2.18}$$

<u>Proof</u>. The scattering equation implies

$$\frac{\Psi^-}{a} = \Phi^+ + \frac{b}{a}e^{2ikx}\Phi^-$$

$$\frac{\Psi^+}{\bar{a}} = \Phi^- + \frac{\bar{b}}{\bar{a}}e^{-2ikx}\Phi^+.$$

Assuming that $a, \bar{a}$ have no zeros in the lower, upper half k-complex plane, the above equations define a Riemann-Hilbert (RH) problem [14], which is equivalent to (2.15). Similarly, the scattering equation in the form $\Phi = \Psi exp(ikx\hat{J})S^{-1}$ implies (2.17). To obtain (2.16) we need to consider the alrge k behavior of (2.15). Equation (2.15a) is

$$\Phi^+(k) = \binom{1}{0} - \frac{1}{2\pi i}⨍_{-\infty}^{\infty} dk' \frac{\frac{b}{a}e^{2ik'x}\Phi^-}{k' - k} - \frac{1}{2}\frac{b}{a}e^{2ikx}\Phi^-(k),$$

where $⨍_{-\infty}^{\infty}$ denotes a principal value integral. As $k \to \infty$, $\frac{b}{a} \to -\frac{r(0)}{ik}$, $\Phi_2^- \to 1$, $\Phi_1^- \to \frac{1}{k}$. The terms with $\frac{1}{k}$ behavior will give a nontrivial contribution:

$$\frac{b}{a}e^{2ik'x}\Phi_2^- = \left(\frac{b}{a}e^{2ik'x}\Phi_2^- + \frac{r(0)}{ik}e^{2ik'x}\right) - \frac{r(0)e^{2ik'x}}{ik'},$$

and

$$⨍_{-\infty}^{\infty} dk' \frac{e^{2ik'x}}{k'(k'-k)} = -\frac{i\pi}{k}\left(1 - e^{2ikx}\right).$$

Hence, for $x > 0$

$$\Phi_2^+(k) \to \frac{1}{2\pi i k}\left[\fint_{-\infty}^{\infty} dk' \left(\frac{b}{a} e^{2ik'x}\Phi_2^- + \frac{r(0)}{ik'} e^{2ik'x}\right) + \pi r(0)(1 - e^{2ikx})\right], \quad \text{as} \quad k \to \infty. \quad (2.19)$$

Comparing equation (2.19) with

$$\Phi_2^+(k) \to -\frac{r(x)}{2ik} + \frac{r(0)e^{2ikx}}{ik},$$

(see equation(2.14)), it follows that

$$r(x) = -\frac{1}{\pi}\fint_{-\infty}^{\infty} dk \left(\frac{b}{a} e^{2ikx}\Phi_2^-(k) + \frac{r(0)}{ik} e^{-2ikx}\right) + r(0). \quad (2.20)$$

The above reduces to (2.16b), since $\fint_{-\infty}^{\infty} dk \frac{e^{-2ikx}}{k} = -i\pi$, if $x > 0$.

Remarks 2.1.

(i) The linear limit of the inversion formulae given by Proposition 2.4, is the sine transform: In the linear limit $a \to 1$, $\Psi \to I$, and

$$\bar{b} \doteq -\int_{-\infty}^{\infty} d\xi \bar{q} \Psi_2^+ e^{2ik\xi}, \quad q(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{\bar{b}}{a} e^{-2ikx}\Psi_1^-,$$

yield the well known sine transform formulae

$$\bar{b} \doteq -2i\int_0^{\infty} d\xi q(\xi) sin 2k\xi, \quad q(x) = \frac{2i}{\pi}\int_0^{\infty} dk \bar{b}(k) sin 2k\xi. \quad (2.21)$$

(ii) The above procedure reconstructs odd potentials as expected. For example,

$$r(-x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{b(k)}{a(k)} e^{-2ikx}\Phi_2^-(-x,k) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{b(-k)}{a(-k)} e^{2ikx}\Phi_2^-(-x,-k)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{b(k)}{\bar{a}(k)} e^{2ikx}\Psi_2^+(x,k) = -r(x).$$

2.2 Evolution of the Scattering Data

We recall the well known [12] Lax pair associated with the NLS equation

$$\varphi_x = ikJ\varphi + Q\varphi, \quad \varphi_t = U\varphi, \quad U \doteq -2ik^2 J - iqrJ - 2kQ - iQ_x J. \quad (2.22)$$

Indeed, the compatibility condition $\varphi_{xt} = \varphi_{tx}$ implies

$$Q_t = -iQ_{xx}J + 2iqrQJ, \quad (2.23)$$

which reduces to the NLS if $r = \pm q*$, where * denotes complex consjugate.

Proposition 2.5. Let Ψ and Q solve (2.7a) and (2.23) respectively. Then $\psi \doteq \Psi exp(ikxJ)$, solves

$$\psi_t = \tilde{U}\psi + 2ik^2\psi J - 4kH(-x)\psi\psi^{-1}(0,t,k)Q(0,t)\psi(0,t,k), \tag{2.24a}$$

where

$$\tilde{U} \doteq -2ik^2 J - i\tilde{q}\tilde{r}J - 2k\tilde{Q} - i\tilde{Q}_x J, \quad Q(0,t) \doteq \begin{pmatrix} 0 & q(0,t) \\ r(0,t) & 0 \end{pmatrix}. \tag{2.24b}$$

Proof.

(i) We first derive the above result using a continuity argument. It is easily shown that equations (2.22) and $\varphi_t = U\varphi + \varphi F$ also imply (2.23) for an arbitrary function $F(x,k)$. To derive (2.24) we choose F to be a discontinuous function of x such that ψ_t is continuous. Let

$$\psi_t = \tilde{U}\psi + \psi F, \quad x > 0;$$

as $x \to +\infty$, $\psi \to exp(ikxJ)$, thus $F = 2ik^2 J$, hence

$$\psi_t = \tilde{U}\psi + 2ik^2\psi J, \quad x > 0.$$

Let

$$\psi_t = \tilde{U}\psi + 2ik^2\psi J + \psi C, \quad x < 0,$$

and fix C by requiring that ϕ_t is continuous at $x = 0$, thus

$$C = -4k\psi^{-1}(0,t,k)Q(0,t)\psi(0,t,k).$$

(ii) Equation ψ satisfies

$$\psi = e^{ikxJ} - \int_x^\infty d\xi e^{ik(x-\xi)J} Q\psi, \quad x > 0$$

$$\psi = e^{ikxJ} - \int_0^\infty d\xi e^{ik(x-\xi)J} Q\psi + \int_x^0 d\psi e^{ik(x-\xi)J} Q(-\xi)\psi, \quad x < 0.$$

Postulate $\psi_t = \tilde{U}\psi + f$, then for $x > 0$

$$\psi_t = -\int_x^\infty d\xi e^{ik(x-\xi)J}(Q_t\psi + Q\psi_t),$$

or

$$U\psi + f = -\int_x^\infty d\xi e^{ik(x-\xi)J}(-iQ_{xx}J + 2iqrQJ + QU)\psi + f,$$

and similarly for $x < 0$. This yields an integral equation for f which implies (2.24).

Remark 2.2. Equation (2.24) and $\psi_x = ikJ\psi + \tilde{Q}\psi$ imply that $\psi_{tx} - \psi_{xt}$ is a distribution, for details see [13].

Using the t-part of the Lax pair it is now straightforward to derive the evolution of the scattering data:

Proposition 2.6. Let S be defined by (2.8) and assume that Q, ψ evolve according to equation (2.23), (2.24) respectively. Then

$$S_t = -2ik^2[J,S] - 4kS\Psi^{-1}(0,t,k)Q(0,t)\Psi(0,t,k), \tag{2.25}$$

i.e.

$$a_t = -4k(a\mu + \bar{b}M^-), \;\; \bar{a}_t = 4k(\bar{a}\mu - bM^+), \tag{2.26}$$

$$b_t = -4ik^2 b - 4k(b\mu + \bar{a}M^-), \;\; \bar{b}_t = 4ik^2\bar{b} + 4k(\bar{b}\mu - aM^+), \tag{2.27}$$

where

$$M(t,k) \doteqdot \Psi^{-1}(0,t,k)Q(0,t)\Psi(0,t,k) = \begin{pmatrix} q\Psi_2^-\Psi_2^+ - r\Psi_1^+\Psi_1^- & q(\Psi_2^+)^2 - r(\Psi_1^+)^2 \\ r(\Psi_1^-)^2 - q(\Psi_2^-)^2 & r\Psi_1^+\Psi_1^- - q\Psi_2^+\Psi_2 \end{pmatrix}(0,t,k) \doteqdot$$

$$\doteqdot \begin{pmatrix} \mu & M^+ \\ M^- & -\mu \end{pmatrix} \tag{2.28}$$

Remarks 2.3.

(i) In the homogeneous case $Q(0,t) = 0$, then

$$a(t,k) = a(0,k), \;\; \bar{a}(t,k) = \bar{a}(0,k), \;\; b(t,k) = b(0,k)e^{-4ik^2t}, \;\; \bar{b}(t,k) = \bar{b}(0,k)e^{4ik^2t}. \tag{2.29}$$

(ii) In the linear limit, $\mu \to 0$, $M^+ \to q$, $M^- \to r$. Thus

$$\bar{b}_t \sim 4ik^2\bar{b} - 4kq(0,t).$$

This is precisely the time evolution of the sine transform (see (2.3) and (2.21)).

(iii) It can be shown that equations (2.26)-(2.27) are invariant under $k \to -k$.

2.3 A Nonlinear Equation for the Scattering Data

The main difficulty associated with the inhomogeneous boundary value problems is the dependence of the evolution of the scattering data on quadratic products of eigenfunctions evaluated at $x = 0$. It seems quite remarkable that it is possible to completely eliminate these products and obtain equations involving only the scattering data:

Proposition 2.7.

The scattering data $b, \bar{b}$ satisfy the following equations:

$$\frac{\bar{b}_t - 4ik^2\bar{b}}{4k} = -q(0,t) + \int_{-\infty}^{\infty} dk' \frac{\frac{\bar{b}(k')}{8k'} - \frac{\bar{b}(k)}{8k}}{k'-k} \frac{\partial}{\partial t} H \ln(1 - b\bar{b})(k'), \tag{2.30}$$

$$\frac{b_t + 4ik^2 b}{4k} = -r(0,t) + \int_{-\infty}^{\infty} dk' \frac{\frac{b(k')}{8k'} - \frac{b(k)}{8k}}{k'-k} \frac{\partial}{\partial t} H \ln(1 - b\bar{b})(k'), \tag{2.31}$$

where H denotes the Hilbert transform, i.e.

$$(Hf)(x) \doteq \frac{1}{\pi} \int_{-\infty}^{\infty} d\xi \frac{f(\xi)}{\xi - x}. \tag{2.32}$$

Having obtained $b, \bar{b}$, then $a, \bar{a}$ followed by solving the Riemann-Hilbert problem

$$a\bar{a} = 1 + b\bar{b}, \quad a, \bar{a} \to 1 \quad \text{as} \quad k \to \infty. \tag{2.33}$$

Proof . Let

$$N(t,k) \doteq \Phi^{-1}(0,t,k) Q(0,t) \Phi(0,t,k) \doteq \begin{pmatrix} \nu & N^- \\ N^+ & -\nu \end{pmatrix}. \tag{2.34}$$

Then

$$M = \Psi^{-1} Q \Psi = S^{-1} \Phi^{-1} Q \Phi S = S^{-1} N S,$$

thus

$$SM = NS. \tag{2.35}$$

The above equation can be written in the following convenient form:

$$\frac{M^+}{\bar{a}} - \frac{N^-}{a} = \frac{\bar{b}}{a}(a\mu + \bar{b}M^-) + \frac{\bar{b}}{\bar{a}}(\bar{a}\mu - bM^+) \doteq A, \tag{2.36a}$$

$$\frac{N^+}{\bar{a}} - \frac{M^-}{a} = \frac{b}{a}(a\mu + \bar{b}M^-) + \frac{b}{\bar{a}}(\bar{a}\mu - bM^+) \doteq B. \tag{2.36b}$$

Equation (2.36a) implies

$$\frac{M^+}{\bar{a}} = q(0,t) + \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{A(k')}{k' - k - i0}. \tag{2.37}$$

We next express $\frac{M^+}{\bar{a}}$ and A in terms of scattering data:

$$A = -\frac{\bar{b}}{4k}\frac{a_t}{a} + \frac{\bar{b}}{4k}\frac{\bar{a}_t}{\bar{a}} = \frac{\bar{b}}{4k}\frac{\partial}{\partial t}(\ln \bar{a} - \ln a), \tag{2.38}$$

where we have used the definition of A and (2.26). Also equation (2.36a) implies

$$-\frac{N^-}{a} = \frac{\bar{b}}{a}(a\mu + \bar{b}M^-) + \bar{b}\mu - (b\bar{b}+1)\frac{M^+}{\bar{a}} = -\frac{\bar{b}}{4k}\frac{a_t}{a} + \frac{\bar{b}_t - 4ik^2\bar{b}}{4k},$$

where we have used (2.26) and (2.27). Thus

$$\frac{M^+}{\bar{a}} = \frac{N^-}{a} + A = \frac{\bar{b}}{4k}\frac{\bar{a}_t}{\bar{a}} - \frac{\bar{b}_t - 4ik^2\bar{b}}{4k}. \tag{2.39}$$

Substituting (2.38), (2.39) in (2.37) we obtain

$$\frac{\bar{b}_t - 4ik^2\bar{b}}{4k} = -q(0,t) + \frac{\bar{b}}{8k}\left(\frac{\bar{a}_t}{\bar{a}} + \frac{a_t}{a}\right) + iH\frac{\bar{b}}{8k}\left(\frac{\bar{a}_t}{\bar{a}} - \frac{a_t}{a}\right).$$

Thus

$$\frac{\bar{b}_t - 4ik^2\bar{b}}{4k} = -q(0,t) + P^+\left(\frac{\bar{b}}{4k}\frac{a_t}{a}\right) - P^-\left(\frac{\bar{b}}{4k}\frac{\bar{a}_t}{\bar{a}}\right), \tag{2.40}$$

where $P^{\pm}$ denote the usual projection operators, i.e.

$$P^{\pm}f \doteq \pm\frac{f}{2} + \frac{1}{2i}Hf. \tag{2.41}$$

Alternatively, using (2.33) the above yields

$$\frac{\bar{b}_t - 4ik^2\bar{b}}{4k} = -q(0,t) + \frac{\bar{b}}{8k}\frac{\partial}{\partial t}\ln(1+b\bar{b}) + iH\frac{\bar{b}}{4k}\frac{\partial}{\partial t}(\ln\bar{a} - \ln a). \tag{2.42}$$

But

$$\frac{\bar{b}}{8k}\frac{\partial}{\partial t}\ln(1+b\bar{b}) = -\frac{\bar{b}}{8k}H\frac{\partial}{\partial t}H\ln(1+b\bar{b}),$$

and

$$\ln\bar{a} - \ln a = -iH(\ln\bar{a} + \ln a) = -iH\ln(1+b\bar{b}),$$

since $\ln\bar{a}$, $\ln a$ are $+,-$ functions respectively. Thus equation (2.42) implies (2.30). Similarly, equation (2.36b) yields

$$\frac{b_t + 4ik^2b}{4k} = -r(0,t) + P^+\left(\frac{b}{4k}\frac{a_t}{a}\right) - P^-\left(\frac{b}{4k}\frac{\bar{a}_t}{\bar{a}}\right), \tag{2.43}$$

which implies (2.31).

Remark 2.4.

(i) In the linear limit equations (2.30), (2.31) reduce to

$$\bar{b}_t - 4ik^2\bar{b} = -4kq(0,t), \quad b_t + 4ik^2 b = -4kr(0,t),$$

which are the time evolution of the sine transform.

(ii) Equations (2.30), (2.31) become linear if $b\bar{b}$ is known.

(iii) The scattering data $\gamma \doteq b/\bar{a}, \quad \bar{\gamma} \doteq \bar{b}/a$ satisfy the following equations:

$$\bar{\gamma}_t - 4ik^2\bar{\gamma} = -4kq(0,t) + 4kP^- \left\{ \frac{\gamma\bar{\gamma}}{1-\gamma\bar{\gamma}} \left(\frac{\bar{\gamma}_t - 4ik^2\bar{\gamma}}{4k} \right) - \frac{i\bar{\gamma}}{1-\gamma\bar{\gamma}} H \left(\frac{\ln(1-\gamma\bar{\gamma})_t}{4k} \right) \right\},$$

$$\gamma_t + 4ik^2\gamma = -4kr(0,t) + 4kP^+ \left\{ \frac{\gamma\bar{\gamma}}{1-\gamma\bar{\gamma}} \left(\frac{\gamma_t - 4ik^2\gamma}{4k} \right) + \frac{i\gamma}{1-\gamma\bar{\gamma}} H \left(\frac{\ln(1-\gamma\bar{\gamma})_t}{4k} \right) \right\},$$

A. The NLS

The NLS

$$iq_t = q_{xx} - 2\sigma|q|^2 q, \quad \sigma = \pm 1 \tag{2.44}$$

corresponds to $r = \sigma q^*$, then $\bar{b}(k,t) = \sigma b^*(k^*,t)$, and $\bar{b}$ satisfies

$$\frac{\bar{b}_t - 4ik^2\bar{b}}{4k} = -4kq(0,t) + \int_{-\infty}^{\infty} dk' \frac{\frac{\bar{b}(k')}{8k'} - \frac{\bar{b}(k)}{8k}}{k'-k} \frac{\partial}{\partial t} H \ln(1-\sigma|\bar{b}|^2)(k'). \tag{2.45}$$

Proposition 2.8. The initial-boundary value problem associated with the NLS equation (2.44), where $q(x,0)$, $q(0,t)$ are given appropriately smooth functions and $q(x,0)$ decays for large x, is solved by

$$q(x,t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk \bar{\gamma}(k,t) e^{-2ikx} \Psi_1^-(x,t,k), \quad \bar{\gamma} \doteq \frac{\bar{b}}{a}, \quad x > 0, \tag{2.46}$$

where $\bar{b}$ solves the nonlinear integrodifferential equation (2.45), a solves (2.33), and $\Psi = (\Psi_1^-, \Psi_2^-)$ solves the linear integral equations

$$\Psi_1^- = 1 - P^-\{\bar{\gamma}^* e^{2ikx} (\Psi_2^-)^*\}, \quad \Psi_2^- = -\sigma P^-\{\bar{\gamma}^* e^{2ikx} (\Psi_1^-)^*\}. \tag{2.47}$$

Proof. The above result follows from Proposition 2.7 and 2.4: for real k, $\gamma = \sigma\bar{\gamma}^*$, thus (2.17) imply $\Psi_1^+ = \sigma(\Psi_2^-)^*$, $\Psi_2^+ = (\Psi_1^-)^*$ and they reduce to (2.47).

Remark 2.5. If $q(0,t) = 0$, (2.45) reduces $\bar{b}_t - 4ik^2\bar{b} = 0$.

B. A Note on the Odd Extension

The above analysis is based on considering an odd extension of the potential q. This has two consequences: (i) The linear limit of the analysis reduces to the sine transform formalism. (ii) The formalism involves only $q(0,t)$ (which is given) and not $q_x(0,t)$. However, the above formalism is nonlinear, since $\bar{\gamma}$ satisfies a nonlinear integrodifferential equation. It is thus natural to ask if there exist an alternative linear formalism. It appears to the author that the odd extension is the only natural one associated with this problem. This is based on the following. Let us consider

$$\Phi_x = ik[J,\Phi] + Q\Phi, \quad 0 \le x < \infty.$$

The eigenfunctions

$$\Psi = I - \int_x^\infty d\xi e^{ik(x-\xi)\hat{J}} Q\Psi, \quad \Phi = e^{ikx\hat{J}} A(k,t) + \int_0^x d\xi e^{ik(x-\xi)\hat{J}} Q\Phi, \tag{2.48}$$

define the RH problem

$$\Phi = \Psi e^{ikx\hat{J}} S, \quad S \doteq A(k,t) + \int_0^\infty d\xi e^{-ik\xi\hat{J}} Q\Phi, \tag{2.49}$$

provided that A_{21}, A_{12} are $+,-$ functions in k. Letting $\varphi = \Phi e^{ikxJ}$ it can be shown that φ_t satisfies

$$\varphi_t = U\varphi + \varphi A^{-1}[A_t + (2ik^2 J + iQ_x(0,t)J + i(qr)(0,t)J + 2kQ(0,t))A]. \tag{2.50}$$

Then the evolution of the scattering data (2.49b) depends on the term in the bracket appearing in (2.50). Thus we need to choose A such that:

(i) A_{21}, A_{12} have proper analyticity properties in k.

(ii) The evolution of the scattering data does not depend on $Q_x(0,t)$. We claim that if

$$A(k,t) \doteq \int_0^\infty d\xi e^{ik\xi J} Q(\xi,t)\varphi(-\xi,t,k), \tag{2.51}$$

then the above two requirements are satisfied. Indeed

$$A = \int_0^\infty d\xi e^{ik\xi J} Q(\xi,t)\Phi(-\xi,t,k)e^{-ik\xi J} = \int_0^\infty d\xi e^{ik\xi\hat{J}} Q(\xi,t)\Phi(-\xi,t,k)$$

has the correct analyticity properties. Also it can be shown that (ii) is fulfilled. However, the eigenfunctions (2.48) with A defined as above are the eigenfunctions (2.7) which follow from an odd extension. Furthermore, it appears that (2.51) is the unique choice satisfying (i), (ii): From the linear limit of the inversion formula, it follows that

$$A = I - \int_0^\infty d\xi e^{ik\xi J} Q(\xi,t)F(\xi,t,k),$$

where $F \to I$ in the linear limit. The choices $F = I$, or $F = A$ contradict (ii) , while $F = \varphi(\xi,t,k)$ contradicts (i).

3 The General Case

Equation (1.2) can be solved by the transform (1.3). The inverse data satisfy

$$\hat{q}_t - 4ik^2\hat{q} = -\frac{4k}{2ik-\alpha}[q_x(0,t) + \alpha q(0,t)]. \tag{3.1}$$

It should be stressed that the factor $f(k)$ appearing in (1.3) is uniquely determined by the requirement that the inverse data depend only of g(t) and not separately on $q(0,t)$, $q_x(0,t)$. Indeed,

$$\hat{q}_t = \int_0^\infty d\xi(e^{2ik\xi} + f(k)e^{-2ik\xi})q_t(\xi) = -i\int_0^\infty d\xi(e^{2ik\xi} + f(k)e^{-2ik\xi})q_{\xi\xi} =$$

$$= 4ik^2\hat{q} + i(1+f)\left[q_x(0,t) - 2ik\frac{(1-f)}{1+f}q(0,t)\right].$$

Thus, if $-2ik\frac{(1-f)}{(1+f)} = \alpha$, then $f = \frac{2ik+\alpha}{2ik-\alpha}$.

3.1 The Nonlinear Analogue of the Transform (1.3)

Motivated by the linear problem we consider the following extension of the potential Q:

$$\tilde{Q}(x,t,k) \doteq Q(x,t)H(x) + F(k)Q(-x,t)H(-x), \quad F(k) = diag(f(k), f(-k)). \tag{3.2}$$

Remarks 3.1.

(i) Suppose that Q satisfies the first member of the AKNS hierarchy, i.e. Q solves (2.23). Then $F(k)Q(-x,t)$ also solves (2.23). (This follows from the fact that $f(k)f(-k) = 1$.)

(ii) The potential $\tilde{Q}$ satisfies the symmetry condition

$$\tilde{Q}(-x,k) = F(-k)\tilde{Q}(x,k) \tag{3.3}$$

A. Analytic Eigenfunctions

Proposition 3.1. Let the matrices Ψ, Φ solve (2.7) where $\tilde{Q}$ is given by (3.2). Let $x > 0$. Then Ψ, Φ:

(i) Solve (2.6), with $\tilde{Q}$ defined by (3.2).

(ii) Satisfy the following symmetry condition

$$\Phi(-x,-k) = A(-k)\Psi(x,k)A(k), \quad A(k) \doteq \begin{pmatrix} -f(k) & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.4}$$

or in component form, if $\Psi \doteq (\Psi^-, \Psi^+), \Phi = (\Phi^+, \Phi^-)$, then

$$\Phi_1^+(-x,-k) = \Psi_1^-(x,k), \quad \Phi_2^+(-x,-k) = -f(k)\Psi_2^-(x,k),$$

$$\Phi_1^-(-x,-k) = -f(-k)\Psi_1^+(x,k), \quad \Phi_2^+(-x,-k) = \Psi_2^+(x,k) \tag{3.5}$$

(iii) Ψ^+, Ψ^- are $+, -$ vectors in the complex k-plane. Φ^+, Φ^- for $\alpha > 0$ are sectionally meromorphic functions in the complex k plane: Φ^+ is analytic in the upper half complex k-plane and has a pole at $\frac{i\alpha}{2}$, if $\alpha > 0$; similarly Φ^- is analytic in the lower half complex k-plane and has a pole at $\frac{-i\alpha}{2}$ if $\alpha > 0$.

Proof. (i) is straightforward. To prove (ii) note that

$$\Psi(-x,-k) = I - \int_{-x}^{\infty} d\xi e^{-ik(-x-\xi)\hat{J}} \tilde{Q}(\xi,-k)\Psi(\xi,-k) =$$

$$= I - \int_{-\infty}^{x} d\xi e^{ik(x-\xi)\hat{J}} F(-k)\tilde{Q}(\xi,k)\Psi(-\xi,-k).$$

Multiply the above matrix by a matrix $C = diag(C_1, C_2)$ and choose C such that $CF(-k)\tilde{Q}(x,k) = -\tilde{Q}(x,k)C$, i.e. $C_1 = -C_2 f(k)$, for example let $C = A$, defined by (3.4). Thus

$$A\Psi(-x,-k) = A + \int_{-\infty}^{x} d\xi e^{ik(x-\xi)\hat{J}} \tilde{Q}(\xi,k)A\Psi(-\xi,-k),$$

or

$$A\Psi(-x,-k)A^{-1} = I + \int_{-\infty}^{x} d\xi e^{ik(x-\xi)\hat{J}} \tilde{Q}(\xi,k)A\Psi(-\xi,-k)A^{-1},$$

and hence (3.4a) follows, since $A^{-1}(k) = A(-k)$.

(iii) Consider (2.7a) with $x > 0$. Then Ψ^+, Ψ^- are $+, -$ functions respectively, since $\tilde{Q}(\xi,k) = Q(\xi)$. Equation (2.7b), for $x > 0$ imply

$$\Phi_1^+ = 1 + \int_0^x d\xi q\Phi_2^+ + \int_{-\infty}^0 d\xi f(k)q(-\xi)\Phi_2^+,$$

$$\Phi_2^+ = \int_0^x d\xi r\Phi_1^+ e^{2ik(x-\xi)} + \int_{-\infty}^0 d\xi f(-k)r(-\xi)\Phi_1^+ e^{2ik(x-\xi)},$$

letting $\xi \to -\xi$ in the integrals over $(-\infty, 0)$ and using $\Phi_2^+(-x,k) = -f(-k)\Psi_2^-(x,-k)$, $\Phi_1^+(-x,k) = \Psi_1^-(x,-k)$ we obtain

$$\Phi_1^+ = 1 - \int_0^\infty d\xi q\Psi_2^+(\xi,-k) + \int_0^x d\xi q\Phi_2^+,$$

$$\Phi_2^+ = f(-k)\int_0^\infty d\xi e^{2ik(x+\xi)} r\Psi_2^-(\xi,-k) + \int_0^x d\xi e^{2ik(x-\xi)} r\Phi_1^+. \tag{3.6}$$

Since $\Psi_1^-(x,k), \Psi_2^-(x,k)$ are $-$ functions it follows that $\Psi_1^-(x,-k)$, $\Psi_2^-(x,-k)$ are $+$ functions. Also $exp(2ikx)$ is a $+$ function since $x > 0$. Thus the forcing of the above integral equations is a function analytic in the upper half k-complex plane with a pole at $2ik + \alpha = 0$ iff $\alpha > 0$. Similarly

$$\Phi_1^- = f(k)\int_0^\infty d\xi e^{-2ik(x+\xi)} q\Psi_2^+(\xi - k) + \int_0^x d\xi e^{-2ik(x-\xi)} q\Phi_2^-,$$

$$\Phi_2^- = 1 - \int_0^\infty d\xi r\Psi_1^+(\xi - k) + \int_0^x d\xi r\Phi_1^-. \tag{3.7}$$

B. The Scattering Equation

Proposition 3.2.

(i) The eigenfunctions Ψ, Φ defined by (2.7), where $\tilde{Q}$ is given by (3.2), are related via (2.8).

(ii) $detS = 1$.

(iii) The scattering data satisfy the following symmetry condition,

$$S(-k) = A(-k)S^{-1}(k)A(k), \tag{3.8}$$

where $A(k)$ is defined in (3.4b), or in component form

$$a(-k) = \bar{a}(k), \quad \bar{b}(-k) = f(-k)\bar{b}(k), \quad b(-k) = f(k)b(k). \tag{3.9}$$

(iv) $\bar{a}, a$ are analytic in the upper, lower half complex k-plane with a pole at $\frac{i\alpha}{2}$, $\frac{-i\alpha}{2}$ iff $\alpha > 0$.

Proof. The derivation of (i), (ii) is similar to that of Proposition (2.2). To derive (iii) use (3.4). To derive (iv), note that

$$\bar{a}(k) = 1 - f(-k)\int_{-\infty}^{0} d\xi r(-\xi)\Psi_1^+(\xi, k) + \int_0^\infty d\xi r(\xi)\Psi_1^+(\xi, k),$$

or

$$\bar{a}(k) = 1 - f(-k)\int_0^\infty d\xi r\Psi_1^+(-\xi, k) + \int_0^\infty d\xi r\Psi_1^+(\xi, k). \tag{3.10}$$

Similarly for $a(k)$.

Remark 3.2.

(i) When $\alpha \to \infty$, $f \to -1$, $A \to I$ and (3.4), (3.8) reduce to $\Phi(-x, -k) = \Psi(x, k)$, $S(-k)S(k) = I$, i.e. to equations (2.11), (2.12).

(ii) When $\alpha \to 0$, i.e. when $q_x(0, t)$ is given, the linear problem is solved by the cosine transform. In this case $f = 1$, $A = J$ and (3.4), (3.8) reduce to

$$\Phi(-x, -k) = J\Psi(x, k)J, \quad a(-k) = \bar{a}(k), \quad \bar{b}(-k) = \bar{b}(k), \quad b(-k) = b(k). \tag{3.11}$$

C. The Inverse Problem

In the case of $\alpha = 0$, the potential $\tilde{Q}$ is continuous at the origin, while Q_x is discontinuous. Hence the scattering data $b, \bar{b}$ behave like $\frac{1}{k^2}$ for large k. Since for large $k, f \to 1$, actually the above behavior is also valid for all finite values of α (the case $\alpha \to \infty$ is different and was considered separately in §2.)

Proposition 3.3. The inverse formulae of Proposition 2.4 are also valid if the scattering data are defined by (2.8), with $\tilde{Q}$ given by (3.2).

Proof. If $\alpha < 0$ the result is straightforward, since all the quantities of interest have the proper analyticity properties. If $\alpha > 0$, these quantities have *removable* singularities and hence the analysis goes through. For example, near $k = \frac{i\alpha}{2}$, $\Phi^+(k) \sim f(-k)$ and $\bar{a}(k) \sim f(-k)$, thus $\frac{\Phi^+}{\bar{a}}$ has a removable singularity.

Remark 3.2. The linear limit of the inversion formulae given by Proposition 3.3, is the transform defined by (1.3): Recall that

$$q = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk e^{-2ikx}\frac{\bar{b}}{a}\Psi_1^-, \quad x > 0; \quad \bar{b} = -\int_{-\infty}^{\infty} d\xi e^{2ik\xi}\tilde{q}\Psi_2^+. \tag{3.12a}$$

The linear limit is straightforward if $a \leq 0$: Ψ_1^-, Ψ_2^+, a, tend to 1 and the above yield

$$\bar{b} = -\int_{-\infty}^{0} d\xi e^{2ik\xi} f(k) q(-\xi) - \int_{0}^{\infty} d\xi e^{2ik\xi} q, \quad q = -\int_{-\infty}^{0} dk e^{2ikx}\bar{b} - \frac{1}{\pi}\int_{0}^{\infty} dk e^{-2ikx}\bar{b},$$

or

$$q = -\frac{1}{\pi}\int_{0}^{\infty} dk \left(e^{-2ikx} + f(-k)e^{2ikx}\right)\bar{b}(k), \quad \bar{b} = -\int_{0}^{\infty} d\xi \left(e^{2ik\xi} + f(k)e^{-2ik\xi}\right) q. \tag{3.12b}$$

If $\alpha > 0$, then $\frac{\Phi^+}{\bar{a}}$, $\frac{\Phi^-}{a}$ develop pole singularities since Φ^+, Φ^- still behave like $f(-k)$, $f(k)$ near $k = \frac{i\alpha}{2}$, $k = -\frac{i\alpha}{2}$, respectively, but $\bar{a}, a \to 1$. The contribution from these singularities is $e^{-\alpha x}C$, C constant, which yields the additional term appearing in (1.3).

3.2 Evolution of the Scattering Data

In analogy with Propositions 2.5, 2.6 we have:

Proposition 3.4. Let Ψ and Q solve (2.7a) and (2.23) respectively, where $\tilde{Q}$ is given by (3.2). The

(i) $\psi \doteqdot \Psi exp(ikxJ)$ solves

$$\psi_t = \tilde{U}\psi + 2ik^2\psi + iH(-x)\psi\psi(0,t)^{-1}J(I+F)(Q_x(0,t) + \alpha Q(0,t))\psi(0,t), \tag{3.13}$$

where $F = diag(f(k), f(-k))$.

(ii) The scattering data S satisfies

$$S_t = [U_0, S] + i\Psi(0,t)^{-1}J(I+F)(Q_x(0,t) + \alpha Q(0,t))\Psi(0,t). \tag{3.14}$$

Proof. The derivation is similar to that of Proposition 2.5, 2.6: If $\psi_t = \tilde{U}\psi + 2ik^2\psi J + H(-x)\psi C$, continuity implies

$$C = \psi(0,t)^{-1}[2k(F-I)Q(0,t) + i(F+I)JQ_x(0,t)]\psi(0,t) =$$

$$= \psi(0,t)^{-1}[iJ(F+I)(Q_x(0,t) + \alpha Q(0,t))]\psi(0,t).$$

As $x \to -\infty$, (3.13) implies (3.14).

Remarks 3.3.

(i) In the linear limit, the evolution of the scattering data reduces to (3.1). For example, one of the components of (3.14) gives

$$\bar{b}_t = 4ik^2\bar{b} + \frac{4k}{2ik-\alpha}\left(q_x(0,t) + \alpha q(0,t)\right). \tag{3.15}$$

Equations (3.12b), (3.15) provide the solution of (2.1) (for $\alpha \leq 0$).

(ii) In the homogeneous case $Q_x(0,t) + \alpha Q(0,t) = 0$ and the scattering data can be found in closed form (see equations (2.28)).

Exploring the analyticity structure of $\Psi(0,t)$ one may again formulate a nonlinear equation for the scattering data similar to that given in 2.3. The study of this nonlinear singular integrodifferential equation will be presented elsewhere.

Acknowledgements

This work was supported in part by the National Science Foundation under Grant Number DMS-8501325 and the Air Force Office of Scientific Research under Grant Number 87-0310. I am grateful to M.J. Ablowitz for several important suggestions. I also acknowledge useful discussions with J. Bona, J. Keller, V. Papageorgiou, P.M. Santini, E.K. Sklyanin and Y. Yortsos. I am grateful to J. Keller and his group at the Department of Mathematics, for their hospitality during my sabbatical leave at Stanford University.

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